

# OPTIMAL HARDY INEQUALITIES IN CONES

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ABSTRACT. Let  $\Omega$  be an open connected cone in  $\mathbb{R}^n$  with vertex at the origin. Assume that the operator

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega^2(x)}$$

is *subcritical* in  $\Omega$ , where  $\delta_\Omega$  is the distance function to the boundary of  $\Omega$  and  $\mu \leq 1/4$ . We show that under some smoothness assumption on  $\Omega$ , the following improved Hardy-type inequality

$$\int_\Omega |\nabla \varphi|^2 dx - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} dx \geq \lambda(\mu) \int_\Omega \frac{|\varphi|^2}{|x|^2} dx \quad \forall \varphi \in C_0^\infty(\Omega),$$

holds true, and the Hardy-weight  $\lambda(\mu)|x|^{-2}$  is optimal in a certain definite sense. The constant  $\lambda(\mu) > 0$  is given explicitly.

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## 1. INTRODUCTION

Let  $P$  be a symmetric second-order linear elliptic operator with real coefficients, defined in a domain  $\Omega$  of  $\mathbb{R}^n$ , and denote by  $q$  its associated quadratic form. Suppose that  $q(\varphi) \geq 0$  for all  $\varphi \in C_0^\infty(\Omega)$ , i.e.  $P$  is *nonnegative* ( $P \geq 0$ ) in  $\Omega$ . Then  $P$  is called *subcritical* in  $\Omega$  if there exists a nontrivial, nonnegative weight  $W$  such that the following Hardy-type inequality holds true

$$(1.1) \quad q(\varphi) \geq \lambda \int_\Omega W(x) |\varphi(x)|^2 dx \quad \forall \varphi \in C_0^\infty(\Omega),$$

where  $\lambda > 0$  is a constant. If  $P \geq 0$  in  $\Omega$  and (1.1) is not true for any  $W \geq 0$ , then  $P$  is called *critical* in  $\Omega$ .

Given a subcritical operator  $P$  in  $\Omega$ , there is a huge convex set of weights  $W \geq 0$  satisfying (1.1). A natural question is to find a weight function  $W$  which is “as large as possible” and satisfies (1.1) (see Agmon [1, Page 6]).

In the paper [13], the authors have constructed a Hardy-weight  $W$ , for a subcritical operator  $P$ , which is *optimal* in a certain definite sense. For symmetric operators the main result of [13] reads as follows.

**Theorem 1.1** ([13, Theorem 2.2]). *Assume that  $P$  is subcritical in  $\Omega$ . Fix a reference point  $x_0 \in \Omega$ , and set  $\Omega^\star := \Omega \setminus \{x_0\}$ . There exists a nonzero nonnegative weight  $W$  satisfying the following properties:*

(a) *Denote by  $\lambda_0 = \lambda_0(P, W, \Omega^\star)$  the largest constant  $\lambda$  satisfying*

$$(1.2) \quad q(\varphi) \geq \lambda \int_{\Omega^\star} W(x) |\varphi(x)|^2 dx \quad \forall \varphi \in C_0^\infty(\Omega^\star).$$

*Then  $\lambda_0 > 0$  and the operator  $P - \lambda_0 W$  is critical in  $\Omega^\star$ ; that is, the inequality*

$$q(\varphi) \geq \int_{\Omega^\star} V(x) |\varphi(x)|^2 dx \quad \forall \varphi \in C_0^\infty(\Omega^\star)$$

is not valid for any  $V \gneq \lambda_0 W$ .

- (b) The constant  $\lambda_0$  is also the best constant for (1.2) with test functions supported in  $\Omega' \subset \Omega$ , where  $\Omega'$  is either the complement of any fixed compact set in  $\Omega$  containing  $x_0$  or any fixed punctured neighborhood of  $x_0$ .
- (c) The operator  $P - \lambda_0 W$  is null-critical in  $\Omega^*$ ; that is, the corresponding Rayleigh-Ritz variational problem

$$(1.3) \quad \inf_{\varphi \in \mathcal{D}_P^{1,2}(\Omega^*)} \frac{q(\varphi)}{\int_{\Omega^*} W(x) |\varphi(x)|^2 dx}$$

admits no minimizer. Here  $\mathcal{D}_P^{1,2}(\Omega^*)$  is the completion of  $C_0^\infty(\Omega^*)$  with respect to the norm  $u \mapsto \sqrt{q(u)}$ .

- (d) If furthermore  $W > 0$  in  $\Omega^*$ , then the spectrum and the essential spectrum of the Friedrichs extension of the operator  $W^{-1}P$  on  $L^2(\Omega^*, W dx)$  are both equal to  $[\lambda_0, \infty)$ .

**Definition 1.2.** A weight function that satisfies properties (a)–(d) is called an *optimal Hardy weight* for the operator  $P$  in  $\Omega$ .

For related spectral results concerning optimal Hardy inequalities see [12].

One may look at a punctured domain  $\Omega^*$  as a noncompact manifold with two ends  $\infty$  and  $x_0$ , where  $\infty$  denotes the ideal point in the one-point compactification of  $\Omega$ . In fact, the results of Theorem 1.1 are valid on such manifolds. In [13, Theorem 11.6], the authors extend Theorem 1.1 and get an optimal Hardy-weight  $W$  in the *entire* domain  $\Omega$ , in the case of *boundary singularities*, where the two singular points of the Hardy-weight are located at  $\partial\Omega \cup \{\infty\}$  and not at  $\infty$  and at an isolated interior point of  $\Omega$  as in Theorem 1.1. The result reads as follows.

**Theorem 1.3** ([13, Theorem 11.6]). *Assume that  $P$  is subcritical in  $\Omega$ . Suppose that the Martin boundary  $\delta\Omega$  of the operator  $P$  in  $\Omega$  is equal to the minimal Martin boundary and is equal to  $\partial\Omega \cup \{\xi_0, \xi_1\}$ , where  $\partial\Omega \setminus \{\xi_0, \xi_1\}$  is assumed to be a regular manifold of dimension  $(n-1)$  without boundary, and the coefficients of  $P$  are locally regular up to  $\partial\Omega \setminus \{\xi_0, \xi_1\}$ .*

*Denote by  $\hat{\Omega}$  the Martin compactification of  $\Omega$ , and assume that there exists a bounded domain  $D \subset \Omega$  such that  $\xi_0$  and  $\xi_1$  belong to two different connected components  $D_0$  and  $D_1$  of  $\hat{\Omega} \setminus \bar{D}$  such that each  $D_j$  is a neighborhood in  $\hat{\Omega}$  of  $\xi_j$ , where  $j = 0, 1$ .*

*Let  $u_0$  and  $u_1$  be the minimal Martin functions at  $\xi_0$  and  $\xi_1$  respectively. Consider the super-solution  $u_{1/2} := (u_0 u_1)^{1/2}$  of the equation  $Pu = 0$  in  $\Omega$ , and assume that*

$$(1.4) \quad \lim_{\substack{x \rightarrow \xi_0 \\ x \in \Omega}} \frac{u_1(x)}{u_0(x)} = \lim_{\substack{x \rightarrow \xi_1 \\ x \in \Omega}} \frac{u_0(x)}{u_1(x)} = 0.$$

*Then the weight  $W := \frac{Pu_{1/2}}{u_{1/2}}$  is an optimal Hardy weight for  $P$  in  $\Omega$ . Moreover, if  $W$  does not vanish on  $\hat{\Omega} \setminus \{\xi_0, \xi_1\}$ , then the spectrum and the essential spectrum of the Friedrichs extension of the operator  $W^{-1}P$  acting on  $L^2(\Omega, W dx)$  is  $[1, \infty)$ .*

The following example illustrates Theorem 1.3 and motivated our present study.

**Example 1.4** ([13, Example 11.1]). Let  $P = P_0 := -\Delta$ , and consider the cone  $\Omega$  with vertex at the origin, given by

$$(1.5) \quad \Omega := \{x \in \mathbb{R}^n \mid r(x) > 0, \omega(x) \in \Sigma\},$$

where  $\Sigma$  is a Lipschitz domain on the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ ,  $n \geq 2$ , and  $(r, \omega)$  denotes the spherical coordinates of  $x$  (i.e.,  $r = |x|$ , and  $\omega = x/|x|$ ). We assume that  $P$  is subcritical in  $\Omega$ .

Let  $\phi$  be the principal eigenfunction of the (Dirichlet) Laplace-Beltrami operator  $-\Delta_S$  on  $\Sigma$  with principal eigenvalue  $\sigma = \lambda_0(-\Delta_S, \mathbf{1}, \Sigma)$  (for the definition of  $\lambda_0$  see (2.1)), and set

$$\gamma_{\pm} := \frac{2 - n \pm \sqrt{(2 - n)^2 + 4\sigma}}{2}.$$

Then the positive harmonic functions

$$u_{\pm}(r, \omega) := r^{\gamma_{\pm}} \phi(\omega)$$

are the Martin kernels at  $\infty$  and  $0$  [29] (see also [5]).

The function

$$u_{1/2} := (u_+ u_-)^{1/2} = r^{(2-n)/2} \phi(\omega)$$

is a supersolution of the equation  $Pu = 0$  in  $\Omega$  (this is the so called *supersolution construction* for  $P$  in  $\Omega$  with the pair  $(u_+, u_-)$ ).

Consequently, the associated Hardy weight is

$$W(x) := \frac{Pu_{1/2}}{u_{1/2}} = \frac{(n-2)^2 + 4\sigma}{4|x|^2},$$

and the corresponding Hardy-type inequality reads as follows

$$(1.6) \quad \int_{\Omega} |\nabla \varphi|^2 dx \geq \frac{(n-2)^2 + 4\sigma}{4} \int_{\Omega} \frac{|\varphi|^2}{|x|^2} dx \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

It follows from Theorem 1.3 that  $W$  is an optimal Hardy-weight. Note that for  $\Sigma = \mathbb{S}^{n-1}$  we obtain the classical Hardy inequality in the punctured space. We also remark that the Hardy-type inequality (1.6) and the *global* optimality of the constant  $(n-2)^2/4 + \sigma$  are not new (cf. [27, 23]).

Let

$$\delta(x) = \delta_{\Omega}(x) := \text{dist}(x, \partial\Omega)$$

be the distance function to the boundary of a domain  $\Omega$ .

The aim of the present paper is to extend the result in Example 1.4 to the case of the Hardy operator

$$P_{\mu} := -\Delta - \frac{\mu}{\delta_{\Omega}^2(x)} \quad \text{in } \Omega,$$

where  $\Omega$  is the cone defined by (1.5), and  $\mu \leq \mu_0 := \lambda_0(-\Delta, \delta_{\Omega}^{-2}, \Omega)$  under the assumption the  $P_{\mu}$  is subcritical in  $\Omega$  (for the definition of  $\lambda_0$ , see (2.1)). In particular, we obtain an explicit expressions for the optimal Hardy weight  $W$  corresponding to the singular points  $0$  and  $\infty$ , for the associate best Hardy constant, and for the corresponding ground state. Note that since the potential  $\delta_{\Omega}^{-2}(x)$  is singular on  $\partial\Omega$ , Theorem 1.3 is not applicable for  $P_{\mu}$  with  $\mu \neq 0$ , and we had to come up with new techniques and ideas to treat this case. For some recent results concerning sharp Hardy inequalities with boundary singularities see [10, 16, 20] and references therein.

The outline of the present paper is as follows. In Section 2 we fix the setting and notations, and introduce some basic definitions. In Section 3 we use an approximation argument to obtain two positive multiplicative solutions of the equation  $P_{\mu}u = 0$  in  $\Omega$  of the form  $u_{\pm}(r, \omega) := r^{\gamma_{\pm}} \theta(\omega)$ , while in Section 4 we use the boundary Harnack principle of A. Ancona [4] and the methods in [22, 29] to get an explicit representation theorem for the positive solutions of the equation  $P_{\mu}u = 0$  in  $\Omega$  that vanish (in the potential theory sense) on  $\partial\Omega \setminus \{0\}$ . The obtained two

linearly independent positive multiplicative solutions are the building blocks of the supersolution construction that is used in Section 5 to prove our main result. In Section 6 we consider a family of Hardy inequalities in the half-space  $\mathbb{R}_+^n$  obtained by S. Filippas, A. Tertikas and J. Tidblom [18], and we obtain, for the appropriate case, the optimality of the corresponding weight.

We conclude the paper in Section 7 by proving a closely related Hardy-type inequality with the best constant for the (nonnegative) operator  $P_\mu$  in  $\Omega$ , where  $\Omega$  is a domain in  $\mathbb{R}^n$  such that  $0 \in \partial\Omega$ , and  $\delta_\Omega$  satisfies (in the weak sense) the linear differential inequality

$$(1.7) \quad -\Delta\delta_\Omega + \frac{n-1+\sqrt{1-4\mu}}{|x|^2}(x \cdot \nabla\delta_\Omega - \delta_\Omega) \geq 0 \quad \text{in } \Omega.$$

Finally, we note that parts of the results of the present paper were announced in [14].

## 2. PRELIMINARIES

In this section we fix our setting and notations, and introduce some basic definitions. We denote  $\mathbb{R}_+ := (0, \infty)$ , and

$$\mathbb{R}_+^n := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}.$$

Throughout the paper  $\Omega$  is a domain in  $\mathbb{R}^n$ , where  $n \geq 2$ . The distance function to the boundary of  $\Omega$  is denoted by  $\delta_\Omega$ . We write  $\Omega' \Subset \Omega$  if  $\Omega$  is open,  $\overline{\Omega'}$  is compact and  $\overline{\Omega'} \subset \Omega$ . By an *exhaustion* of  $\Omega$  we mean a sequence  $\{\Omega_k\}$  of smooth, relatively compact domains such that  $x_0 \in \Omega_1$ ,  $\Omega_k \Subset \Omega_{k+1}$ , and  $\bigcup_{N=1}^\infty \Omega_k = \Omega$ .

Let  $f, g : \Omega \rightarrow [0, \infty)$ . We denote  $f \asymp g$  in  $\Omega$  if there exists a positive constant  $C$  such that  $C^{-1}g \leq f \leq Cg$  in  $\Omega$ . Also, we write  $f \not\geq 0$  in  $\Omega$  if  $f \geq 0$  in  $\Omega$  but  $f \neq 0$  in  $\Omega$ . We denote by  $\mathbf{1}$  the constant function taking the value 1 in  $\Omega$ .  $B_r(x)$  is the open ball of radius  $r$  centered at  $x$ . If  $\Omega$  is a cone and  $R > 0$ , we denote by  $A_R$  the annulus

$$A_R := \{z \in \Omega \mid \frac{R}{2} \leq |z| \leq 2R\}.$$

In the present paper we consider a second-order linear elliptic operator  $P$  defined on a domain  $\Omega \subset \mathbb{R}^n$ , and let  $W \not\geq 0$  be a given function. We write  $P \geq 0$  in  $\Omega$  if the equation  $Pu = 0$  in  $\Omega$  admits a positive (super)solution. Unless otherwise stated it is assumed that  $P \geq 0$  in  $\Omega$ .

Throughout the paper it is assumed that the operator  $P$  is *symmetric* and locally uniformly elliptic. Moreover, we assume that coefficients of  $P$  and the function  $W$  are real valued and locally sufficiently regular in  $\Omega$  (see [13]). For such an operator  $P$ , potential  $W$ , and  $\lambda \in \mathbb{R}$ , we denote  $P_\lambda := P - \lambda W$ .

The following well known Agmon-Allegretto-Piepenbrink (AAP) theorem holds (see for example [2] and references therein).

**Theorem 2.1** (The AAP Theorem). *Suppose that  $P$  is symmetric, and let  $q$  be the corresponding quadratic form. Then  $P \geq 0$  in  $\Omega$  if and only if  $q(\varphi) \geq 0$  for every  $\varphi \in C_0^\infty(\Omega)$ .*

We recall the following definitions.

**Definition 2.2.** Let  $q$  be the quadratic form on  $C_0^\infty(\Omega)$  associated with a symmetric nonnegative operator  $P$  in  $\Omega$ . We say that a sequence  $\{\varphi_k\} \subset C_0^\infty(\Omega)$  of nonnegative functions is a *null-sequence* of the quadratic form  $q$  in  $\Omega$ , if there exists an open set  $B \Subset \Omega$  such that

$$\lim_{k \rightarrow \infty} q(\varphi_k) = 0, \quad \text{and} \quad \int_B |\varphi_k|^2 dx = 1.$$

We say that a positive function  $\phi \in C_{\text{loc}}^\alpha(\Omega)$  is a (*Agmon*) *ground state* of the functional  $q$  in  $\Omega$  if  $\phi$  is an  $L_{\text{loc}}^2(\Omega)$  limit of a null-sequence of  $q$  in  $\Omega$ .

**Definition 2.3.** Let  $K \Subset \Omega$ , and let  $u$  be a positive solution of the equation  $Pw = 0$  in  $\Omega \setminus K$ . We say that  $u$  is a *positive solution of minimal growth in a neighborhood of infinity* in  $\Omega$  if for any  $K \Subset K' \Subset \Omega$  with smooth boundary and any (regular) positive supersolution  $v \in C((\Omega \setminus K') \cup \partial K')$  of the equation  $Pw = 0$  in  $\Omega \setminus K'$  satisfying  $u \leq v$  on  $\partial K'$ , we have  $u \leq v$  in  $\Omega \setminus K'$ .

**Theorem 2.4** ([31]). *Suppose that  $P$  is nonnegative symmetric operator in  $\Omega$ , and let  $q$  be the corresponding quadratic form. Then the following assertions are equivalent*

- (i) *The operator  $P$  is critical in  $\Omega$ .*
- (ii) *The quadratic form admits a null-sequence and a ground state  $\phi$  in  $\Omega$ .*
- (iii) *The equation  $Pu = 0$  admits a unique positive supersolution  $\phi$  in  $\Omega$ .*
- (iv) *The equation  $Pu = 0$  admits a positive solution in  $\Omega$  of minimal growth in a neighborhood of infinity in  $\Omega$ .*

*In particular, any ground state is the unique positive (super)solution of the equation  $Pu = 0$  in  $\Omega$ , and it has minimal growth in a neighborhood of  $\infty$ .*

Let  $P$  and  $W \not\equiv 0$  be as above, the *generalized principal eigenvalue* is defined by

$$(2.1) \quad \lambda_0 := \lambda_0(P, W, \Omega) := \sup \left\{ \lambda \in \mathbb{R} \mid P_\lambda = P - \lambda W \geq 0 \text{ in } \Omega \right\}.$$

We also define

$$\lambda_\infty = \lambda_\infty(P, W, \Omega) := \sup \left\{ \lambda \in \mathbb{R} \mid \exists K \subset\subset \Omega \text{ s.t. } P_\lambda \geq 0 \text{ in } \Omega \setminus K \right\}.$$

Recall that if the operator  $P$  is symmetric in  $L^2(\Omega, dx)$ , and  $W > 0$ , then  $\lambda_0$  (resp.  $\lambda_\infty$ ) is the infimum of the  $L^2(\Omega, Wdx)$ -spectrum (resp.  $L^2(\Omega, Wdx)$ -essential spectrum) of the Friedrichs extension of the operator  $\tilde{P} := W^{-1}P$  (see for example [2] and references therein). Note that  $\tilde{P}$  is symmetric on  $L^2(\Omega, Wdx)$ , and has the same quadratic form as  $P$ .

**Definition 2.5.** Let  $\Omega \subsetneq \mathbb{R}^n$  be a domain. We say that  $\Omega$  is *weakly mean convex* if  $\delta_\Omega$  is weakly superharmonic in  $\Omega$ .

Recall that  $\delta_\Omega \in W_{\text{loc}}^{1,2}(\Omega)$ . Also, any convex domain is of course weakly mean convex, and if  $\partial\Omega \in C^2$ , then  $\Omega$  is weakly mean convex if and only if the mean curvature at any point of  $\partial\Omega$  is nonnegative (see for example [33]).

Throughout the paper we fix a cone

$$(2.2) \quad \Omega := \{x \in \mathbb{R}^n \mid r(x) > 0, \omega(x) \in \Sigma\},$$

where  $\Sigma$  is a Lipschitz domain in the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ ,  $n \geq 2$ . For  $x \in \Sigma$ , we will denote  $d_\Sigma(x)$  the (spherical) distance from  $x$  to the boundary of  $\Sigma$ . Note that  $\delta_\Omega$  is clearly a homogeneous function of degree 1, that is,

$$(2.3) \quad \delta_\Omega(x) = |x| \delta_\Omega\left(\frac{x}{|x|}\right) = r \delta_\Omega(\omega).$$

Since the distance function to the boundary of any domain is Lipschitz continuous, Euler's homogeneous function theorem implies that

$$(2.4) \quad x \cdot \nabla \delta_\Omega(x) = \delta_\Omega(x) \quad \text{a.e. in } \Omega.$$

In fact, Euler's theorem characterizes all sufficiently smooth positive homogeneous functions. Hence, (2.4) characterizes the cones in  $\mathbb{R}^n$ . For spectral results and Hardy inequalities with homogeneous weights on  $\mathbb{R}^n$  see [21].

We note that if  $\Sigma$  is  $C^2$ , then

$$(2.5) \quad \delta_\Omega(\omega) = \sin(d_\Sigma(\omega)) \quad \text{near the boundary of } \Sigma.$$

Indeed, for  $\omega \in \Sigma$ , let  $z \in \partial\Omega$  such that  $|z - \omega| = \delta_\Omega(\omega)$ , and let  $y \in \partial\Sigma$  realizes  $d_\Sigma(\omega)$ . Since  $\Sigma$  is  $C^2$ , if  $\omega$  is close enough to  $\partial\Sigma$ , then  $z$  is unique and  $\neq 0$ , and the points  $0, z, y$  are collinear. Moreover, the acute angle between the vectors  $\vec{0y}$  and  $\vec{0\omega}$  is equal to  $d_\Sigma(\omega)$ . Given that  $\vec{0z}$  is orthogonal to  $\vec{\omega z}$ , by elementary trigonometry in the triangle  $0, \omega, y$ , one gets that  $\delta_\Omega(\omega) = \sin(d_\Sigma(\omega))$ .

Let  $\Delta_S$  be the Laplace-Beltrami operator on the unit sphere  $S := \mathbb{S}^{n-1}$ . Then in spherical coordinates, the operator

$$P_\mu := -\Delta - \frac{\mu}{\delta_\Omega^2}$$

has the following skew-product form

$$(2.6) \quad P_\mu u(r, \omega) = -\frac{\partial^2 u}{\partial r^2} - \frac{n-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left( -\Delta_S u - \mu \frac{u}{\delta_\Omega^2(\omega)} \right) \quad r > 0, \omega \in \Sigma.$$

For any Lipschitz cone the Hardy inequality holds true (as in the case of sufficiently smooth bounded domain [24]). We have

**Lemma 2.6.** *Let  $\Omega$  be a Lipschitz cone, and let  $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$ . Then*

$$(2.7) \quad 0 < \mu_0 \leq \frac{1}{4}.$$

*In other words, the following Hardy inequality holds true.*

$$(2.8) \quad \int_\Omega |\nabla \varphi|^2 dx \geq \mu_0 \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} dx \quad \forall \varphi \in C_0^\infty(\Omega),$$

where  $0 < \mu_0 \leq 1/4$  is the best constant.

Moreover, if  $\Omega$  is a weakly mean convex domain, then  $\mu_0 = 1/4$ .

*Proof.* Using Rademacher's theorem it follows that  $\partial\Omega$  admits a tangent hyperplane almost everywhere in  $\partial\Omega$ . Hence, [24, Theorem 5] implies that

$$\mu_0 = \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega) \leq \lambda_\infty(-\Delta, \delta_\Omega^{-2}, \Omega) \leq \frac{1}{4}.$$

We claim that  $\mu_0 > 0$ . Indeed, denote by  $\Omega_R$  the truncated cone

$$(2.9) \quad \Omega_R := \{x \in \mathbb{R}^n \mid 0 < r < R, \omega \in \Sigma\},$$

then

$$0 < \lambda_{0,R} := \lambda_0(-\Delta, \delta_{\Omega_R}^{-2}, \Omega_R),$$

(see for example, [28, 24]). By comparison,

$$\mu_0 \leq \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R), \quad \text{and} \quad 0 < \lambda_{0,R} \leq \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R).$$

It is well known that if  $\{\Omega_k\}$  is an exhaustion of  $\Omega$ , then

$$\lim_{k \rightarrow \infty} \lambda_0(P, W, \Omega_k) = \lambda_0(P, W, \Omega).$$

Hence,

$$\lim_{R \rightarrow \infty} \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) = \mu_0.$$

On the other hand, since  $\delta_\Omega$  is homogeneous of order 1, it follows that  $\lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R)$  is  $R$ -independent. Therefore,

$$0 < \lambda_{0,1} \leq \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_1) = \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) = \lim_{R \rightarrow \infty} \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) = \mu_0.$$

Consequently,

$$\mu_0 = \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega_R) > 0.$$

Assume further that  $\Omega$  is a convex cone, or even a weakly mean convex cone. Then it is well known that  $\mu_0 = 1/4$  (see for example [7, 24]).  $\blacksquare$

**Remark 2.7.** Clearly,  $P_\mu$  is subcritical in  $\Omega$  for all  $\mu < \mu_0$ , and by Proposition 5.8,  $P_{1/4}$  is subcritical in a weakly mean convex cone. We show in Theorem 5.6 that if  $\mu_0 < 1/4$  and  $\Sigma \in C^2$ , then the operator  $P_{\mu_0}$  is critical in the cone  $\Omega$  (cf. [24, Theorem II]).

### 3. POSITIVE MULTIPLICATIVE SOLUTIONS

As above, let  $\Omega$  be a Lipschitz cone. By Lemma 2.6 the generalized principal eigenvalue  $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$  satisfies  $0 < \mu_0 \leq 1/4$ . We have

**Theorem 3.1.** *Let  $\mu \leq \mu_0$ . Then the equation  $P_\mu u = 0$  in  $\Omega$  admits positive solutions of the form*

$$(3.1) \quad u_\pm(x) = |x|^{\gamma_\pm} \phi_\mu\left(\frac{x}{|x|}\right),$$

where  $\phi_\mu$  is a positive solution of the equation

$$(3.2) \quad \left(-\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)}\right) \phi_\mu = \sigma(\mu) \phi_\mu \quad \text{in } \Sigma,$$

$$(3.3) \quad -\frac{(n-2)^2}{4} \leq \sigma(\mu) := \lambda_0\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2}, \mathbf{1}, \Sigma\right),$$

and

$$(3.4) \quad \gamma_\pm := \frac{2-n \pm \sqrt{(n-2)^2 + 4\sigma(\mu)}}{2}.$$

Moreover, if  $\sigma(\mu) > -(n-2)^2/4$ , then there are two linearly independent positive solutions of the equation  $P_\mu u = 0$  in  $\Omega$  of the form (3.1), and  $P_\mu$  is subcritical in  $\Omega$ .

In particular, for any  $\mu \leq \mu_0$  we have  $\sigma(\mu) > -\infty$ .

*Proof.* We first note that if  $u$  is a positive solution of the form (3.1), then clearly  $\phi_\mu > 0$  and  $\phi_\mu$  solves (3.2), and  $\gamma_\pm$  satisfies (3.4).

Fix a reference point  $x_1 \in \Omega \cap \mathbb{S}^{n-1}$ , and consider an *exhaustion*  $\{\Sigma_k\}_{k=1}^\infty \subset \Sigma \subset \mathbb{S}^{n-1}$  of  $\Sigma$  (i.e.,  $\{\Sigma_k\}_{k=1}^\infty$  is a sequence of smooth, relatively compact domains in  $\Sigma$  such that  $x_1 \in \Sigma_k \Subset \Sigma_{k+1}$  for  $k \geq 1$ , and  $\cup_{k=1}^\infty \Sigma_k = \Sigma$ ).

Fix  $\mu \leq \mu_0$ . For  $k \geq 1$ , and denote the cone

$$\mathcal{W}_k := \{x \in \mathbb{R}^n \mid r > 0, \omega \in \Sigma_k\}.$$

Consider the convex set  $\mathcal{K}_{P_\mu}^0(\mathcal{W}_k)$  of all positive solutions  $u$  of the equation  $P_\mu u = 0$  in  $\mathcal{W}_k$  satisfying the Dirichlet boundary condition  $u = 0$  on  $\partial\mathcal{W}_k \setminus \{0\}$ , and the normalization condition  $u(x_1) = 1$ .

Clearly, for  $\mu \leq \mu_0$  we have

$$\mu \leq \lambda_0(-\Delta, \delta_\Omega^{-2}, \mathcal{W}_k) = \sup \{\lambda \in \mathbb{R} \mid \mathcal{K}_{P_\mu}^0(\mathcal{W}_k) \neq \emptyset\}.$$



Moreover,  $P_\mu$  is subcritical in  $\mathcal{W}_k$ , and has Fuchsian-type singularities at the origin and at infinity. Hence, in view of [29, Theorem 7.1], it follows that  $\mathcal{K}_{P_\mu}^0(\mathcal{W}_k)$ , which is a convex compact set in the compact-open topology, has exactly two extreme points.

Next, we characterize the two extreme points of  $\mathcal{K}_{P_\mu}^0(\mathcal{W}_k)$  using two different approaches.

**First method:** We use the results of Section 8 of [22]. Consider the multiplicative group  $\mathcal{G} := \mathbb{R}^*$  of all positive real numbers. Then  $\mathcal{G}$  acts on  $\overline{\mathcal{W}_k} \setminus \{0\}$  (and also on  $\overline{\Omega} \setminus \{0\}$ ) by homotheties  $x \mapsto sx$ , where  $s \in \mathcal{G}$  and  $x \in \overline{\mathcal{W}_k} \setminus \{0\}$ . This is a compactly generating (cocompact) abelian group action, and  $P_\mu$  is an invariant elliptic operator with respect to this action on  $\mathcal{W}_k$ . In spherical coordinates, a positive  $\mathcal{G}$ -multiplicative function on  $\mathcal{W}_k$  is of the form

$$(3.5) \quad f(r, \omega) = r^\gamma \phi(\omega),$$

where  $\gamma \in \mathbb{R}$ . We note that positive solutions in  $\mathcal{K}_{P_\mu}^0(\mathcal{W}_k)$  satisfy a uniform boundary Harnack principle on  $\partial\mathcal{W}_k \setminus \{0\}$ . Recall that  $\mathcal{K}_{P_\mu}^0(\mathcal{W}_k)$  has exactly two extreme points. Hence, by theorems 8.7 and 8.8 of [22],  $\lambda_0(-\Delta, \delta_\Omega^{-2}, \mathcal{W}_k) > \mu$ , and the two extreme points in  $\mathcal{K}_{P_\mu}^0(\mathcal{W}_k)$  are positive  $\mathcal{G}$ -multiplicative solutions of the equation  $P_\mu u = 0$  in  $\mathcal{W}_k$ , and therefore, they have the form

$$(3.6) \quad u_{\pm,k}(r, \omega) = r^{\gamma_{\pm,k}} \phi_{\pm,k}(\omega).$$

In particular,  $\phi_{\pm,k}$  vanish on  $\Sigma_k$ .

Using the spherical coordinates representation (2.6) of  $P_\mu$ , it follows, that  $\phi_{\pm,k}$  are positive in  $\Sigma$ , satisfy  $\phi_{\pm,k}(x_1) = 1$ , and solve the eigenvalue Dirichlet problem

$$(3.7) \quad \left( -\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)} \right) \phi_{\pm,k} = (\gamma_{\pm,k}^2 + \gamma_{\pm,k}(n-2)) \phi_{\pm,k} \text{ in } \Sigma_k, \quad \phi_{\pm,k} = 0 \text{ on } \partial\Sigma_k.$$

On the other hand, since the operator  $-\Delta_S - \mu\delta_\Omega^{-2}$  has up to the boundary regular coefficients in  $\Sigma_k$ , it admits a unique (Dirichlet) eigenvalue  $\sigma_k$  with a positive eigenfunction  $\phi_k$  satisfying  $\phi_k(x_1) = 1$ . Moreover,  $\sigma_k$  is simple. In other words,  $\sigma_k$  and  $\phi_k$  are respectively the *principal* eigenvalue and eigenfunction of  $-\Delta_S - \mu\delta_\Omega^{-2}$  in  $\Sigma_k$ .

Hence,  $\phi_{\pm,k}$  are equal to  $\phi_k$ , and

$$\sigma_k := \sigma_k(\mu) = (\gamma_{\pm,k}^2 + \gamma_{\pm,k}(n-2)).$$

By the strict monotonicity with respect to bounded domains of the principal eigenvalue of second-order elliptic operators with up to the boundary regular coefficients, it follows that  $\sigma_k(\mu) > \sigma_{k+1}(\mu)$ .

On the other hand, since

$$(3.8) \quad u_{\pm,k}(r, \omega) = r^{\gamma_{\pm,k}} \phi_k(\omega) > 0,$$

it follows that  $\gamma_{-,k} \neq \gamma_{+,k}$ , and  $\gamma_{\pm,k}$  are given by

$$\gamma_{\pm,k} := \frac{2 - n \pm \sqrt{(n-2)^2 + 4\sigma_k}}{2}.$$

In particular,

$$\gamma_{-,k} < \gamma_{-,k+1} < \frac{2-n}{2} < \gamma_{+,k+1} < \gamma_{+,k} \quad \text{and} \quad \sigma_k > -\frac{(n-2)^2}{4}.$$

**Second method:** We only indicate briefly the second approach. We use the results of [26]. By (2.6), the subcritical elliptic operator  $P_\mu$  has a skew-product form in  $\mathcal{W}_k = \mathbb{R}_+ \times \Sigma_k$  and satisfies the conditions of Theorem 1.1 of [26]. Therefore, the equation  $P_\mu u = 0$  admits two Martin functions of the form (3.6).



Now, let  $k \rightarrow \infty$ . Then  $\sigma_k \searrow \sigma \geq -(n-2)^2/4$ , and up to a subsequence  $\phi_k \rightarrow \phi_\mu$  locally uniformly in  $\Sigma$ . Clearly,  $\sigma$  does not depend on the exhaustion of  $\Sigma$ . Recall also that for any nonnegative second-order elliptic operator  $L$  in a domain  $D$  and any exhaustion  $\{D_k\}$  of  $D$  we have

$$\lambda_0(L, W, D) = \lim_{k \rightarrow \infty} \lambda_0(L, W, D_k).$$

Hence,  $\sigma = \sigma(\mu) = \lambda_0(-\Delta_S - \mu\delta_\Omega^{-2}, \mathbf{1}, \Sigma)$ .

Consequently,  $\gamma_{\pm, k} \rightarrow \gamma_\pm$ , where  $\gamma_- \leq -(n-2)/2 \leq \gamma_+$ . Hence, we have that

$$\lim_{k \rightarrow \infty} u_{\pm, k}(r, \omega) = \lim_{k \rightarrow \infty} r^{\gamma_{\pm, k}} \phi_k(\omega) = r^{\gamma_\pm} \phi_\mu(\omega).$$

If  $\gamma_- < -(n-2)/2 < \gamma_+$  (or equivalently,  $\sigma(\mu) > -(n-2)^2/4$ ), then we obtain two linearly independent  $\mathcal{G}$ -multiplicative positive solutions of the equation  $P_\mu u = 0$  in  $\Omega$ . In particular,  $P_\mu$  is subcritical in  $\Omega$ .  $\blacksquare$

**Remark 3.2.** Note that for  $n = 2$ ,  $\Sigma = \mathbb{S}^1$ , and  $\mu = \mu_0 = 0$ , we obtain  $\sigma(0) = 0$ ,  $\gamma_\pm = 0$ , and  $P_0 = -\Delta$  is critical in the cone  $\mathbb{R}^2 \setminus \{0\}$ .

**Remark 3.3.** Let  $\Sigma$  be a bounded domain in a smooth Riemannian manifold  $M$ , and let  $d_\Sigma$  be the Riemannian distance function to the boundary  $\partial\Sigma$ . If  $\Sigma$  is smooth enough, then the Hardy inequality with respect to the weight  $(d_\Sigma)^{-2}$  holds in  $\Sigma$  with a positive constant  $C_H$  [34]. A sufficient condition for the validity of a such Hardy inequality is that  $\Sigma$  is *boundary distance regular*, and this condition holds true if  $\Sigma$  satisfies either the *uniform interior cone condition* or the *uniform exterior ball condition* (see the definitions in [34]). For other sufficient conditions for the validity of the Hardy inequality on Riemannian manifolds see for example [25].

Hence, if the cone  $\Omega \subsetneq \mathbb{R}^n \setminus \{0\}$  is smooth enough, then  $\Sigma \subset \mathbb{S}^{n-1}$  is boundary distance regular. So, for such  $\Sigma \subset \mathbb{S}^{n-1}$ , there exists  $C > 0$  such that  $-\Delta_S - C d_\Sigma^{-2} \geq 0$  in  $\Sigma$ . Note that  $d_\Sigma(\omega) \asymp \delta_\Omega(\omega)|_\Sigma$  in  $\Sigma$ , therefore,  $-\Delta_S - C_1 \delta_\Omega^{-2} \geq 0$  in  $\Sigma$  for some  $C_1 > 0$ .

In the sequel we shall need the following lemma concerning the criticality of the operator  $\mathcal{L}_\mu := -\Delta_S - \mu\delta_\Omega^{-2} - \sigma(\mu)$  in  $\Sigma$ .

**Lemma 3.4.** *Consider the operator  $\mathcal{L}_\mu = -\Delta_S - \mu\delta_\Omega^{-2} - \sigma(\mu)$  on  $\Sigma$ . Then*

(1) *We have*

$$(3.9) \quad \mu_0 = \lambda_0\left(-\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma\right).$$

- (2) *Assume that  $\Sigma \in C^2$ , and  $\mu_0 < 1/4$ . Then  $\sigma(\mu_0) = -(n-2)^2/4$ , and  $\mathcal{L}_{\mu_0}$  is critical in  $\Sigma$  with ground state  $\phi_{\mu_0} \in L^2(\Sigma, \delta_\Omega^{-2} dS)$ .*
- (3) *Assume that  $\Sigma \in C^2$ , and  $\mu_0 = 1/4$ . Then  $\mathcal{L}_{1/4}$  is critical in  $\Sigma$  with ground state  $\phi_{1/4} \in L^2(\Sigma, \delta_\Omega^{-2} \log(\delta_\Omega)^{-(1+\epsilon)} dS)$ , where  $\epsilon$  is any positive number.*
- (4) *Assume that  $\mu < \mu_0$ , then  $\mathcal{L}_\mu$  is positive critical in  $\Sigma$ . That is,  $\mathcal{L}_\mu$  admits a ground state  $\phi_\mu$  in  $\Sigma$ , and  $\phi_\mu \in L^2(\Sigma)$ .*

*In particular, in all the above cases,  $\phi_\mu$  is (up to a multiplicative constant) the unique positive (super)solution of the equation  $\mathcal{L}_\mu u = 0$  in  $\Sigma$ , and  $\phi_\mu \in L^2(\Sigma)$ .*

*Proof.* 1. To prove (3.9) we note that Theorem 3.1 implies that for  $\mu \leq \mu_0$  there exists  $\phi_\mu$  positive solution of

$$\mathcal{L}_\mu u = \left(-\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu)\right)u = 0 \quad \text{in } \Sigma,$$

and since for any  $\mu \leq \mu_0$ , we have  $\sigma(\mu) \geq -(n-2)^2/4$ , it follows that  $\phi_\mu$  is a positive supersolution of the equation

$$\mathcal{L}_\mu u = \left( -\Delta_S - \frac{\mu}{\delta_\Omega^2} + \frac{(n-2)^2}{4} \right) u = 0 \quad \text{in } \Sigma.$$

Thus, by the AAP Theorem (Theorem 2.1) we get,

$$\mu_0 \leq \lambda_0 \left( -\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma \right).$$

Let us now take  $\mu > \mu_0$ , and assume by contradiction that  $-\Delta_S + (n-2)^2/4 - \mu\delta_\Omega^{-2} \geq 0$  in  $\Sigma$ . Then by definition, there is a positive solution  $\phi_\mu$  of the equation

$$\left( -\Delta_S - \frac{\mu}{\delta_\Omega^2} + \frac{(n-2)^2}{4} \right) u = 0 \quad \text{in } \Sigma.$$

If one defines

$$\psi(x) = |x|^{(2-n)/2} \phi_\mu \left( \frac{x}{|x|} \right),$$

then it is immediate to check that  $\psi$  is a positive solution in  $\Omega$  of

$$\left( -\Delta - \frac{\mu}{\delta_\Omega^2} \right) u = 0 \quad \text{in } \Omega.$$

This implies that

$$\lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega) \geq \mu > \mu_0,$$

a contradiction. Thus, the operator  $-\Delta_S + (n-2)^2/4 - \mu\delta_\Omega^{-2}$  cannot be nonnegative in  $\Sigma$  for  $\mu > \mu_0$ , and this implies that

$$\mu_0 \geq \lambda_0 \left( -\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma \right).$$

Hence, (3.9) is proved.

2. Since

$$d_\Sigma(x) \sim \delta_\Omega(x) \quad \text{as } x \in \Sigma, d_\Sigma(x) \rightarrow 0,$$

and in light of the proof of [24, Theorem 5], our assumption that  $\Sigma$  is  $C^2$  implies that

$$\lambda_\infty(-\Delta_S, \delta_\Omega^{-2}, \Sigma) = \frac{1}{4},$$

which in turn implies that

$$\lambda_\infty \left( -\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma \right) = \frac{1}{4}.$$

On the other hand, by part 1 we have

$$\lambda_0 \left( -\Delta_S + \frac{(n-2)^2}{4}, \delta_\Omega^{-2}, \Sigma \right) = \mu_0.$$

Hence, our assumption that  $\mu_0 < 1/4$ , implies that there is a spectral gap between the bottom of the  $L^2(\Sigma, \delta_\Omega^{-2} dS)$ -spectrum and the bottom of the essential spectrum of the operator  $-\Delta_S + (n-2)^2/4$  in  $\Sigma$ . Consequently, the operator  $-\Delta_S + (n-2)^2/4 - \mu_0\delta_\Omega^{-2}$  is critical in  $\Sigma$ , with ground state  $\phi_{\mu_0} \in L^2(\Sigma, \delta_\Omega^{-2} dS)$ . Clearly, the criticality of  $-\Delta_S + (n-2)^2/4 - \mu_0\delta_\Omega^{-2}$  in  $\Sigma$  implies that

$$\sigma(\mu_0) = -\frac{(n-2)^2}{4},$$

and the second part of the lemma is proved.

Before proving part 3, we prove the fourth part of the lemma.

4. The assumption  $\mu < \mu_0$  clearly implies that  $\lambda_\infty(-\Delta_S - \mu\delta_\Omega^{-2}, \mathbf{1}, \Sigma) = \infty$ . Hence,

$$-\frac{(n-2)^2}{4} \leq \sigma(\mu) = \lambda_0\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2}, \mathbf{1}, \Sigma\right) < \lambda_\infty(-\Delta_S - \mu\delta_\Omega^{-2}, \mathbf{1}, \Sigma) = \infty.$$

Since  $\lambda_0$  (respect.  $\lambda_\infty$ ) is the bottom of the (respect. essential)  $L^2$ -spectrum of the operator  $-\Delta_S - \mu\delta_\Omega^{-2}$  in  $\Sigma$ , it follows that the operator  $\mathcal{L}_\mu$  is critical in  $\Sigma$ , and  $\sigma(\mu)$  is the principal eigenvalue of the operator  $-\Delta_S - \mu\delta_\Omega^{-2}$  with principal eigenfunction  $\phi_\mu \in L^2(\Sigma)$ . Hence, the operator  $\mathcal{L}_\mu$  is positive critical in  $\Sigma$ .

3. The proof uses a modification of Agmon's trick ([3, Theorem 2.7], see also [24, Lemma 7]). In order to prove that  $\lambda_\infty(-\Delta_S - 1/(4\delta_\Omega^2), \mathbf{1}, \Sigma) = \infty$ , we will show that for suitable positive constants  $c, \varepsilon$ , the function  $\delta_\Omega^{1/2} - \delta_\Omega/2$  is a positive supersolution of the equation

$$(3.10) \quad \left(-\Delta_S - \frac{1}{4\delta_\Omega^2} - \frac{c}{\delta_\Omega^\varepsilon}\right)u = 0$$

in a sufficiently small neighborhood of the boundary of  $\Sigma$ .

We start by denoting a tubular neighborhood of  $\partial\Sigma$  having width  $\beta > 0$ , by

$$\Sigma_\beta := \{\omega \in \Sigma \mid d_\Sigma(\omega) < \beta\}.$$

Recall that since  $\Sigma$  is  $C^2$ , there exists  $\beta_* > 0$  such that  $d_\Sigma \in C^2$  in  $\Sigma_{\beta_*}$ . In particular,  $-\Delta_S d_\Sigma$  is bounded on  $\Sigma_{\beta_*}$ . Also  $|\nabla_S d_\Sigma| = 1$  and  $\delta_\Omega = \sin(d_\Sigma)$  (by (2.5)), both on  $\Sigma_{\beta_*}$ . We may thus compute

$$-\Delta_S \delta_\Omega = \cos(d_\Sigma) \Delta_S d_\Sigma - \sin(d_\Sigma) \quad \text{on } \Sigma_{\beta_*},$$

which implies that  $\Delta_S \delta_\Omega$  is also bounded on  $\Sigma_{\beta_*}$ . In particular, we have

$$(3.11) \quad -\Delta_S \delta_\Omega(\omega) \geq -h \quad \text{for all } \omega \in \Sigma_{\beta_*},$$

for some  $h > 0$ . Now let  $c, \varepsilon > 0$  and compute on  $\Sigma_{\beta_*}$

$$\begin{aligned} & \left(-\Delta_S - \frac{1}{4\delta_\Omega^2} - \frac{c}{\delta_\Omega^\varepsilon}\right) \left(\delta_\Omega^{1/2} - \frac{\delta_\Omega}{2}\right) \\ &= -\frac{1}{4\delta_\Omega^{3/2}}(1 - |\nabla_S \delta_\Omega|^2) - \frac{1}{2\delta_\Omega^{1/2}}(1 - \delta_\Omega^{1/2})\Delta_S \delta_\Omega + \frac{1}{8\delta_\Omega} - c\delta_\Omega^{1/2-\varepsilon} + \frac{c}{2}\delta_\Omega^{1-\varepsilon} \\ &\geq -\frac{\beta_*^2}{4\delta_\Omega^{1/2}} - \frac{h}{2\delta_\Omega^{1/2}}(1 - \delta_\Omega^{1/2}) + \frac{1}{8\delta_\Omega} - c\delta_\Omega^{1/2-\varepsilon} + \frac{c}{2}\delta_\Omega^{1-\varepsilon}, \end{aligned}$$

where we have used the fact that  $1 - |\nabla_S \delta_\Omega|^2 = \sin^2(d_\Sigma) \leq \beta_*^2$  on  $\Sigma_{\beta_*}$  and also (3.11). Clearly, by fixing  $\varepsilon$  in  $(0, 3/2)$  we obtain that this estimate blows up as  $\omega \in \Sigma_{\beta_*}$  approaches the boundary of  $\Sigma$ . Thus, for a smaller  $\beta_* > 0$  if necessary, we proved that  $\delta_\Omega^{1/2} - \delta_\Omega/2$  is a positive supersolution of (3.10) in  $\Sigma_{\beta_*}$ . The APP theorem (Theorem 2.1) implies

$$(3.12) \quad \int_{\Sigma_{\beta_*}} \left(|\nabla u|^2 - \frac{1}{4\delta_\Omega^2}\right) \varphi^2 dS \geq c \int_{\Sigma_{\beta_*}} \frac{\varphi^2}{\delta_\Omega^\varepsilon} dS \quad \forall \varphi \in C_0^\infty(\Sigma_{\beta_*}),$$

which together with  $\lim_{d_\Sigma(\omega) \rightarrow 0} \delta_\Omega^{-\varepsilon}(\omega) = \infty$  imply that

$$\lambda_\infty\left(\Delta_S - \frac{1}{4\delta_\Omega^2}, \mathbf{1}, \Sigma\right) = \infty.$$

As in the proof of part 2, one concludes that  $\mathcal{L} = \Delta_S - 1/(4\delta_\Omega^2) - \sigma(\mu)$  is critical, with ground state  $\phi_{1/4} \in L^2(\Sigma)$ .

It remains to show that in fact,  $\phi_{1/4} \in L^2(\Sigma, \delta_\Omega^{-2} \log^{-(1+\epsilon)}(\delta_\Omega) dS)$ . In fact, the arguments used in the proof of [24, Lemma 9] show that, as  $\omega \in \Sigma$  and  $\delta_\Omega(\omega) \rightarrow 0$ ,

$$\phi_{1/4}(\omega) \asymp \delta_\Omega^{1/2}(\omega).$$

This implies that  $\phi_{1/4} \in L^2(\Sigma, \delta_\Omega^{-2} \log^{-(1+\epsilon)}(\delta_\Omega) dS)$  for any  $\epsilon > 0$ . ■

**Proposition 3.5.** *Let  $\sigma(\mu) = \lambda_0(-\Delta_S - \mu\delta_\Omega^{-2}, \mathbf{1}, \Sigma)$ . Then*

- (1)  $\sigma(\mu) \geq -(n-2)^2/4$  for any  $\mu \leq \mu_0$ , and if  $\Sigma \in C^2$  and  $\mu_0 < 1/4$ , then  $\sigma(\mu_0) = -(n-2)^2/4$ .
- (2)  $\sigma(\mu) = -\infty$  for any  $\mu > 1/4$ .
- (3) If  $\Sigma \in C^2$ , then  $\sigma(\mu) > -\infty$  for all  $\mu \leq 1/4$ .

*Proof.* 1. Recall that by Lemma 2.6 we have that  $0 < \mu_0 \leq 1/4$ , and by Theorem 3.1  $\sigma(\mu) \geq -(n-2)^2/4$  for all  $\mu \leq \mu_0$ . Moreover, by Lemma 3.4, if  $\Sigma \in C^2$  and  $\mu_0 < 1/4$ , then  $\sigma(\mu_0) = -(n-2)^2/4$ . In particular, for such  $\mu$  we have that  $\sigma(\mu)$  is finite.

2. Let  $\mu > 1/4$ , and suppose that  $\sigma(\mu)$  is finite. Then one can find a positive function  $\phi$  satisfying

$$(-\Delta_S - \mu\delta_\Omega^{-2} - \sigma(\mu))\phi = 0 \quad \text{in } \Sigma.$$

Take  $\varepsilon > 0$  such that  $\mu - \varepsilon > 1/4$ . Clearly,

$$\lim_{\omega \rightarrow \partial\Sigma} \delta_\Omega^{-2}(\omega) = \infty, \quad \text{and} \quad \lim_{\omega \rightarrow \partial\Sigma} \frac{\delta_\Omega(\omega)}{d_\Sigma(x)} = 1,$$

where  $d_\Sigma$  is the Riemannian distance to the boundary of  $\Sigma$ . Hence,  $\phi$  is a positive supersolution of the equation

$$(-\Delta_S - (\mu - \varepsilon)d_\Sigma^{-2})u = 0$$

in a neighborhood of infinity in  $\Sigma$ .

On the other hand, as in [24], if  $\Sigma$  is a Lipschitz domain, then  $\lambda_\infty(-\Delta_S, d_\Sigma^{-2}, \Sigma) \leq 1/4$ . Consequently, for such  $\varepsilon$ , one gets a contradiction to  $\lambda_\infty(-\Delta_S, d_\Sigma^{-2}, \Sigma) \leq 1/4$ .

3. Suppose first that  $\mu < 1/4$ . Recall that since  $\Sigma \in C^2$  we have

$$\lambda_\infty(-\Delta_S, \delta_\Omega^{-2}, \Sigma) = \lambda_\infty(-\Delta_S, d_\Sigma^{-2}, \Sigma) = 1/4.$$

Take  $\varepsilon > 0$  such that  $\mu + \varepsilon < 1/4$ . Let  $\phi$  be a positive solution of the equation

$$(-\Delta_S - (\mu + \varepsilon)\delta_\Omega^{-2})u = 0$$

in a neighborhood of infinity in  $\Sigma$ , and let  $\tilde{\phi}$  be a nice positive function in  $\Sigma$  such that  $\tilde{\phi} = \phi$  in a neighborhood of  $\partial\Sigma$ . Then for  $\sigma$  large enough,  $\tilde{\phi}$  is a positive supersolution of the equation  $(-\Delta_S - \mu\delta_\Omega^{-2} + \sigma)u = 0$  in  $\Sigma$ . Hence  $\sigma(\mu) > -\infty$  for all  $\mu < 1/4$ .

Suppose now that  $\mu = 1/4$ . By (3.10),  $\psi := \delta_\Omega^{1/2} - \delta_\Omega/2$  is a positive supersolution of

$$\left(-\Delta_S - \frac{1}{4\delta_\Omega^2} - \frac{c}{\delta_\Omega^\varepsilon}\right)u = 0$$

outside a compact set  $K_\varepsilon \Subset \Sigma$ . Let  $\tilde{\psi}$  be a nice positive function in  $\Sigma$  such that  $\tilde{\psi} = \psi$  in a neighborhood of  $\partial\Sigma$ . Hence, for  $\sigma$  large enough,  $\tilde{\psi}$  is a positive supersolution of the equation  $(-\Delta_S - 1/4\delta_\Omega^2 + \sigma)u = 0$  in  $\Sigma$ . Hence  $\sigma(1/4) > -\infty$ .  $\blacksquare$

**Remark 3.6.** In Lemma 3.4 and Proposition 3.5, it is assumed that  $\Sigma \in C^2$ . The extension of the proposition to the class of Lipschitz domains remains open. We recall that by the recent result of G. Barbatis and P. D. Lamberti [8, Proposition 1], the Hardy constant of a bounded domain is Lipschitz continuous as a function of bi-Lipschitz maps that approximate the domain. It seems that finding for a given Lipschitz domain a uniform bi-Lipschitz smooth approximation is a nontrivial problem: we note that in [11, Theorem 1], the authors prove the existence of approximation of Lipschitz homeomorphisms by smooth ones in the  $W^{1,p}$  topology for  $p < \infty$ . However, to apply the results in [8], we should need  $W^{1,\infty}$ -approximations.

We conclude the present section with the following general result that provides us with a sufficient condition for the criticality of a Schrödinger operator on a precompact domain. For a general sufficient condition see [30].

**Lemma 3.7.** *Let  $P = -\Delta + V$  be a nonnegative Schrödinger operator on a compact Riemannian manifold with boundary  $M$ , endowed with its Riemannian measure  $dx$ . Denote by  $\delta = \delta_M$  the distance function to the boundary of  $M$ . Assume that  $M \in C^2$ ,  $V$  is smooth in the interior of  $M$ , and that the equation  $Pu = 0$  in  $M$  admits a positive solution  $\phi \in L^2(M, \delta^{-2} \log^{-2}(\delta) dx)$ . Then,  $P$  is critical in  $M$  with ground state  $\phi$ , and furthermore, there exists a null-sequence  $\{\phi_k\}_{k=0}^\infty$  for  $P$ , which converges locally uniformly and in  $L^2$  to  $\phi$ .*

*Proof.* If  $q$  denotes the quadratic form of  $P$ , then using the ground state transform (see for example [13]) we have for every  $\varphi \in C_0^\infty(M)$ ,

$$q(\phi\varphi) = \int_M \phi^2 |\nabla \varphi|^2 dx.$$

This formula extends easily to every Lipschitz continuous function  $\varphi$  which is compactly supported in  $M$ . For  $k \geq 2$ , let us define  $v_k : \mathbb{R}_+ \rightarrow [0, 1]$  by

$$v_k(t) = \begin{cases} 0 & 0 \leq t \leq 1/k^2, \\ 1 + \frac{\log(kt)}{\log k} & 1/k^2 < t < 1/k, \\ 1 & t \geq 1/k. \end{cases}$$

Note that  $0 \leq v_k(\delta) \leq 1$ , and  $\{v_k(\delta)\}_{k \geq 2}$  converges pointwise to the constant function  $\mathbf{1}$  in  $M$ . Define

$$\phi_k := v_k(\delta)\phi,$$

then, using that  $\phi \in L_{\text{loc}}^2$ , one sees that  $\{\phi_k\}_{k=0}^\infty$  converges locally uniformly and hence in  $L_{\text{loc}}^2$  to  $\phi$ . We now prove that  $\{\phi_k\}_{k=2}^\infty$  is a null-sequence for  $P$ , which implies that  $P$  is critical with ground state  $\phi$ . If  $K \Subset M$  is a fixed precompact open set, then clearly, there is a positive constant  $C$  such that, for  $k$  big enough,

$$\int_K \phi_k^2 dx \asymp 1.$$

Thus, in order to prove that  $\{\phi_k\}_{k=2}^\infty$  is a null-sequence for  $P$ , it is enough to prove that

$$(3.13) \quad \lim_{k \rightarrow \infty} \int_M \phi^2 |\nabla v_k(\delta)|^2 dx = 0.$$

Since  $|\nabla\delta(x)| \leq 1$  a.e. in  $M$ , it is enough to show that

$$\lim_{k \rightarrow \infty} \int_M \phi^2 |v'_k(\delta)|^2 dx = 0.$$

We compute

$$\int_M \phi^2 |v'_k(\delta)|^2 dx = \int_{\{1/k^2 < \delta < 1/k\}} \left( \frac{\phi}{\delta \log(k)} \right)^2 dx \leq 4 \int_{\{\delta < 1/k\}} \left( \frac{\phi}{\delta \log(\delta)} \right)^2 dx.$$

By our hypothesis, the function  $\phi^2 \delta^{-2} \log^{-2}(\delta)$  is integrable on  $\{\delta < 1/2\}$ , hence,

$$\lim_{k \rightarrow \infty} \int_{\{\delta < 1/k\}} \left( \frac{\phi}{\delta \log(\delta)} \right)^2 dx = 0,$$

which shows (3.13). Thus,  $\{\phi_k\}_{k \geq 2}$  is a null-sequence for  $P$ . ■

#### 4. THE STRUCTURE OF $\mathcal{K}_{P_\mu}^0(\Omega)$

As above, let  $\Omega$  be a Lipschitz cone. By Lemma 2.6 the generalized principal eigenvalue  $\mu_0 := \lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega)$  satisfies  $0 < \mu_0 \leq 1/4$ .

For  $\mu \leq \mu_0$ , denote by  $\mathcal{K}_{P_\mu}^0(\Omega)$  the convex set of all positive solutions  $u$  of the equation  $P_\mu u = 0$  in  $\Omega$  satisfying the normalization condition  $u(x_1) = 1$ , and the Dirichlet boundary condition  $u = 0$  on  $\partial\Omega \setminus \{0\}$  in the sense of the Martin boundary. That is, any  $u \in \mathcal{K}_{P_\mu}^0(\Omega)$  has minimal growth on  $\partial\Omega \setminus \{0\}$ . For the definition of minimal growth on a portion  $\Gamma$  of  $\partial\Omega$ , see [29].

If  $\mu_0 < 1/4$  and  $\Sigma$  is  $C^2$ , then in Theorem 5.6 (to be proved in the sequel) we show that the operator  $P_{\mu_0}$  is critical in  $\Omega$ , and therefore the equation  $P_{\mu_0} u = 0$  in  $\Omega$  admits (up to a multiplicative constant) a unique positive supersolution. Moreover, by Theorem 3.1, the unique positive solution is a multiplicative solution of the form (3.1).

The following theorem characterizes the structure of  $u \in \mathcal{K}_{P_\mu}^0(\Omega)$  for any  $\mu < \mu_0$ .

**Theorem 4.1.** *Let  $\mu < \mu_0 \leq 1/4$ . Then  $\mathcal{K}_{P_\mu}^0(\Omega)$  is the convex hull of two linearly independent positive solutions of the equation  $P_\mu u = 0$  in  $\Omega$  of the form*

$$(4.1) \quad u_\pm(x) = |x|^{\gamma_\pm} \phi_\mu \left( \frac{x}{|x|} \right),$$

where  $\phi_\mu$  is the unique positive solution of the equation

$$(4.2) \quad \left( -\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)} \right) \phi_\mu = \sigma(\mu) \phi_\mu \quad \text{in } \Sigma,$$

$$(4.3) \quad -\frac{(n-2)^2}{4} < \sigma(\mu) := \lambda_0 \left( -\Delta_S - \frac{\mu}{\delta_\Omega^2}, \mathbf{1}, \Sigma \right), \text{ and}$$

$$(4.4) \quad \gamma_\pm := \frac{2-n \pm \sqrt{(2-n)^2 + 4\sigma(\mu)}}{2}.$$

*Proof.* The assumption  $\mu < \mu_0$  implies that the operator  $P_\mu$  is subcritical in  $\Omega$ . In particular,  $\mu < 1/4$ , and therefore, there exists  $\varepsilon > 0$  such that the operator  $P_{\mu+\varepsilon}$  is subcritical in a small neighborhood of a given portion of  $\partial\Omega \setminus \{0\}$ . Since the operator  $P_\mu$  and the cone  $\Omega$  are invariant under scaling, it follows from the local Harnack inequality, and from the boundary Harnack principle of A. Ancona for the operator  $P_\mu$  in  $\Omega$  [4] (see also [6]) that the following

uniform boundary Harnack principle holds true in the annulus  $A_R \subset \Omega$ . There exists  $C > 0$  (independent of  $R$ ) such that

$$(4.5) \quad C^{-1} \frac{v(x)}{v(y)} \leq C^{-1} \frac{u(x)}{u(y)} \leq C \frac{v(x)}{v(y)} \quad \forall x, y \in A_R,$$

for any  $u, v \in \mathcal{K}_{P_\mu}^0(\Omega)$  and  $R > 0$ .

Hence, we can use directly the arguments in [29] to obtain that in the subcritical case the convex set  $\mathcal{K}_{P_\mu}^0(\Omega)$  has exactly two extreme points. Moreover, we can use directly the method of [22, Section 8], to obtain that  $u$  is an extreme point of  $\mathcal{K}_{P_\mu}^0(\Omega)$  if and only if it is a positive multiplicative solution in  $\mathcal{K}_{P_\mu}^0(\Omega)$ . Thus, the two extreme points of  $\mathcal{K}_{P_\mu}^0(\Omega)$  are of the form

$$u_\pm(x) = |x|^{\gamma_\pm} \phi_\pm\left(\frac{x}{|x|}\right),$$

where  $\phi_\pm > 0$  in  $\Sigma$ , and solves the equation

$$(4.6) \quad \left(-\Delta_S - \frac{\mu}{\delta_\Omega^2(\omega)}\right) \phi_\pm = \sigma_\pm \phi_\pm \quad \text{in } \Sigma,$$

$$(4.7) \quad -\frac{(n-2)^2}{4} \leq \sigma_\pm \leq \sigma(\mu) := \lambda_0\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2}, \mathbf{1}, \Sigma\right), \text{ and}$$

$$(4.8) \quad \gamma_\pm := \frac{2-n \pm \sqrt{(n-2)^2 + 4\sigma_\pm}}{2}.$$

If  $\gamma_+ = \gamma_-$ , then (4.5) implies that  $u_+ \asymp u_-$ . Since  $u_\pm(x)$  are two extreme points, and  $\mathcal{K}_{P_\mu}^0(\Omega)$  has exactly two extreme points, it follows that  $\gamma_+ \neq \gamma_-$ . Therefore,  $\sigma_\pm = \sigma$ , where  $-(n-2)^2/4 < \sigma \leq \sigma(\mu)$  and  $\gamma_\pm$  satisfy

$$(4.9) \quad \gamma_\pm := \frac{2-n \pm \sqrt{(n-2)^2 + 4\sigma}}{2}.$$

Moreover, since  $\phi_\pm$  solve the same equation in  $\Sigma$ , and  $\mathcal{K}_{P_\mu}^0(\Omega)$  has exactly two extreme points, it follows that  $\phi_\pm = \phi$ .

Note that by Lemma 3.4,  $\phi$  is a positive solution of minimal growth near  $\partial\Sigma$  if and only if  $\sigma = \sigma(\mu)$ . On the other hand,  $u_\pm$  have minimal growth near  $\partial\Omega \setminus \{0\}$ . Therefore,  $\phi = \phi_\mu$  and  $\sigma = \sigma(\mu)$ , where  $\phi_\mu$  is a ground state satisfying (4.2), and  $\sigma(\mu)$  and  $\gamma_\pm$  satisfy (4.3) and (4.4), respectively.  $\blacksquare$

## 5. THE MAIN RESULT

The present section is devoted to our main result concerning the existence of an optimal Hardy weight for the operator  $P_\mu$  which is defined in a cone  $\Omega$ . In Theorem 5.4 we prove the case where  $\mu < \mu_0$  and  $\Omega$  is a Lipschitz cone, while in Theorem 5.6 we prove the case  $\mu = \mu_0$  under the assumption that  $\Sigma \in C^2$ .

Let us recall that by Theorem 3.1, if  $\mu \leq \mu_0$ , then

$$\sigma(\mu) := \lambda_0\left(-\Delta - \frac{\mu}{\delta_\Omega^2}, \mathbf{1}, \Sigma\right) \geq -\frac{(n-2)^2}{4},$$

and there exists a positive solution  $\phi_\mu$  of the equation

$$\left(-\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu)\right)u = 0 \quad \text{in } \Sigma.$$



Furthermore, by Lemma 3.4, the operator

$$\mathcal{L} := \mathcal{L}_\mu = -\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu)$$

is critical (for any  $\mu < \mu_0$ , and also for  $\mu = \mu_0$  if in addition  $\Sigma \in C^2$ ), and  $\phi_\mu$  is the ground state of  $\mathcal{L}$ .

We first prove.

**Proposition 5.1.** *Let  $\Omega$  be a Lipschitz cone. Let  $\mu \leq \mu_0$ , and let*

$$(5.1) \quad \lambda(\mu) := \frac{(2-n)^2 + 4\sigma(\mu)}{4}.$$

*Then  $\lambda(\mu) \geq 0$ , and the following Hardy inequality holds true in  $\Omega$ :*

$$(5.2) \quad \int_\Omega |\nabla \varphi|^2 dx - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} dx \geq \lambda(\mu) \int_\Omega \frac{|\varphi|^2}{|x|^2} dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

*Proof.* The fact that  $\lambda(\mu) \geq 0$  follows from  $\sigma(\mu) \geq -(n-2)^2/4$ , which has been proved in Theorem 3.1. Define

$$\psi(x) = |x|^{(2-n)/2} \phi_\mu\left(\frac{x}{|x|}\right).$$

Then, taking into account that

$$\left(-\Delta_S - \sigma(\mu) - \frac{\mu}{\delta_\Omega^2}\right) \phi_\mu = 0 \quad \text{in } \Sigma,$$

and writing  $P_\mu$  in spherical coordinates (2.6), it follows that  $\psi$  is a positive solution of the equation

$$(P_\mu - \lambda(\mu)|x|^{-2})u = 0 \quad \text{in } \Omega.$$

Thus, the operator  $P_\mu - \lambda(\mu)|x|^{-2}$  is nonnegative in  $\Omega$ , and so (5.2) holds by the AAP Theorem (Theorem 2.1).  $\blacksquare$

**Remark 5.2.** In the case  $\mu < \mu_0$ , the Hardy inequality (5.2) can be obtained using the *supersolution construction* of [13]: indeed, by Theorem 4.1, the equation  $P_\mu u = 0$  has two linearly independent, positive solutions in  $\Omega$ , of the form

$$u_\pm(x) = |x|^{\gamma_\pm} \phi_\mu\left(\frac{x}{|x|}\right).$$

By the *supersolution construction* ([13, Lemma 5.1]), the positive function

$$u_{1/2} := (u_+ u_-)^{1/2} = |x|^{(2-n)/2} \phi_\mu\left(\frac{x}{|x|}\right)$$

is a solution of

$$\left(P_\mu - \frac{|\nabla(u_+/u_-)|^2}{4(u_+/u_-)^2}\right)u = 0 \quad \text{in } \Omega.$$

It is easy to check that

$$\frac{|\nabla(u_+/u_-)|^2}{4(u_+/u_-)^2} = \frac{\lambda(\mu)}{|x|^2},$$

and by the AAP theorem, the Hardy inequality (5.2) holds.

**Remark 5.3.** In the case  $\mu \leq \mu_0$ , the Hardy inequality (5.2) can also be obtained using spherical coordinates, Fubini's theorem, and the well-known one-dimensional Hardy-inequality

$$(5.3) \quad \int_0^\infty (v')^2 t^{n-1} dt \geq \left(\frac{n-2}{2}\right)^2 \int_0^\infty v^2 t^{n-3} dt,$$

valid for all functions  $v \in H^1(\mathbb{R}_+)$  that vanish at  $\infty$ , one easily obtains (5.2) for any  $\mu \in \mathbb{R}$ .

Indeed, suppose that  $\varphi \in C_c^\infty(\Omega)$ . Then we have that  $\varphi_{\Sigma_r}$ , the restriction of  $\varphi$  on  $\Sigma_r$ , is in  $C_c^\infty(\Sigma)$ . Consequently, by the definition of  $\sigma(\mu)$ , it follows that for all  $\varphi \in C_c^\infty(\Omega)$  and each  $r > 0$  we have

$$\int_{\Sigma_r} |\nabla_\omega \varphi|^2 dS_r - \mu \int_{\Sigma_r} \frac{\varphi^2}{\delta_\Omega^2(\omega)} dS_r \geq \sigma(\mu) \int_{\Sigma_r} \varphi^2 dS_r.$$

Multiplying this by  $r^{-2}$  and integrating in  $\mathbb{R}_+$  with respect to  $r$ , we arrive at

$$\int_0^\infty \int_{\Sigma_r} \frac{|\nabla_\omega \varphi|^2}{r^2} dS_r dr - \mu \int_0^\infty \int_{\Sigma_r} \frac{\varphi^2}{r^2 \delta_\Omega^2(\omega)} dS_r dr \geq \sigma(\mu) \int_0^\infty \int_{\Sigma_r} \frac{\varphi^2}{r^2} dS_r dr.$$

Recall that in spherical coordinates we have

$$|\nabla \varphi|^2 = \frac{|\nabla_\omega \varphi|^2}{r^2} + \varphi_r^2,$$

and taking into account (2.3), the last inequality is written as follows

$$\int_\Omega |\nabla \varphi|^2 dx - \mu \int_\Omega \frac{\varphi^2}{\delta_\Omega^2(x)} dx \geq \sigma(\mu) \int_\Omega \frac{\varphi^2}{|x|^2} dx + \int_\Sigma \int_0^\infty \varphi_r^2 r^{n-1} dr dS,$$

where we have used Fubini's theorem on the last term. Applying (5.3) in the inner integral of the last term and using Fubini's theorem again, we obtain (5.2) for any  $\mu \in \mathbb{R}$ .

We now investigate the optimality of the Hardy inequality (5.2) when  $\mu < \mu_0$ .

**Theorem 5.4.** *Let  $\Omega$  be a Lipschitz cone, and let  $\mu < \mu_0$ . Then  $\lambda(\mu) > 0$ . Furthermore the weight  $W := \lambda(\mu)|x|^{-2}$  is an optimal Hardy weight for the operator  $P_\mu$  in  $\Omega$  in the following sense:*

- (1) *The operator  $P_\mu - \lambda(\mu)|x|^{-2}$  is critical in  $\Omega$ , i.e., the Hardy inequality*

$$\int_\Omega |\nabla \varphi|^2 dx - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} dx \geq \int_\Omega V(x) |\varphi|^2 dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

*holds true for  $V \geq W$  if and only if  $V = W$ . In particular,*

$$\lambda_0\left(P_\mu, \frac{1}{|x|^2}, \Omega\right) = \lambda(\mu).$$

- (2) *The constant  $\lambda(\underline{\mu})$  is also the best constant for (5.2) with test functions supported either in  $\Omega_R$  or in  $\Omega \setminus \Omega_R$ , where  $\Omega_R$  is a fixed truncated cone of the form (2.9). In particular,*

$$\lambda_\infty\left(P_\mu, \frac{1}{|x|^2}, \Omega\right) = \lambda(\mu).$$

- (3) *The operator  $P_\mu - \lambda(\mu)|x|^{-2}$  is null-critical at 0 and at infinity in the following sense: For any  $R > 0$  the (Agmon) ground state of the operator  $P_\mu - \lambda(\mu)|x|^{-2}$  given by*

$$v(x) := |x|^{(2-n)/2} \phi_\mu\left(\frac{x}{|x|}\right)$$

satisfies

$$\int_{\Omega_R} \left( |\nabla v|^2 - \mu \frac{|v|^2}{\delta_\Omega^2} \right) dx = \int_{\Omega \setminus \overline{\Omega_R}} \left( |\nabla v|^2 - \mu \frac{|v|^2}{\delta_\Omega^2} \right) dx = \infty.$$

In particular, the variational problem

$$\inf_{\varphi \in \mathcal{D}_{P_\mu}^{1,2}(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2 dx - \mu \int_\Omega \frac{|\varphi|^2}{\delta_\Omega^2} dx}{\int_\Omega \frac{|\varphi|^2}{|x|^2} dx}$$

does not admit a minimizer.

- (4) The spectrum and the essential spectrum of the Friedrichs extension of the operator  $W^{-1}P_\mu = \lambda(\mu)^{-1}|x|^2P_\mu$  on  $L^2(\Omega, W dx)$  are both equal to  $[1, \infty)$ .

**Remark 5.5.** As is pointed out in Remark 5.2, if  $\mu < \mu_0$ , then the Hardy inequality (5.2) can be obtained by applying the *supersolution construction* from [13]. Thus, Theorem 5.4 extends Theorem 1.1 to the particular singular case, where  $\Omega$  is a cone and  $P_\mu$  is the Hardy operator (which is singular on  $\partial\Omega$ ).

*Proof of Theorem 5.4.* In light of our assumption that  $\mu < \mu_0 \leq 1/4$ , it follows the operator  $P_\mu$  is subcritical in  $\Omega$ . Moreover, by Theorem 4.1,  $\sigma(\mu) > -(n-2)^2/4$ , so  $\lambda(\mu) > 0$ . For such a  $\mu$ , consider the operator  $\mathcal{L} = \mathcal{L}_\mu$  on  $\Sigma \subset \mathbb{S}^{n-1}$  defined by

$$\mathcal{L} = -\Delta_S - \frac{\mu}{\delta_\Omega^2} - \sigma(\mu),$$

with the corresponding nonnegative quadratic form

$$q_{\mathcal{L}}(\psi) = \int_\Sigma \left( |\nabla_\omega \psi|^2 - \mu \frac{|\psi|^2}{\delta_\Omega^2} - \sigma(\mu) |\psi|^2 \right) dS \quad \text{where } \psi \in C_0^\infty(\Sigma).$$

Notice that by Lemma 3.4,  $\mathcal{L}$  is critical in  $\Sigma$  with the ground state  $\phi_\mu \in L^2(\Sigma)$ . We normalize  $\phi_\mu$  so that  $\int_\Sigma \phi_\mu^2 dS = 1$ .

On the other hand, it is well known that the operator

$$\mathcal{R} := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r} \frac{\partial}{\partial r} - \frac{(n-2)^2}{4r^2}$$

is critical on  $\mathbb{R}_+$ , and  $r^{(2-n)/2}$  is its ground state. Indeed, the corresponding quadratic form  $q_{\mathcal{R}}$  of  $\mathcal{R}$  (endowed with the measure  $r^{n-1} dr$ ) is given by

$$q_{\mathcal{R}}(u) = \int_0^\infty \left[ (u')^2 - \frac{(n-2)^2}{4} \frac{u^2}{r^2} \right] r^{n-1} dr \quad u \in C_0^\infty(\mathbb{R}_+),$$

and gives rise to the critical operator  $\mathcal{R}$  on  $\mathbb{R}_+$ .

Recall that in spherical coordinates  $P_\mu - W$  has the following skew-product form:

$$P_\mu - W = \mathcal{R} \otimes \mathcal{I}_\Sigma - \frac{\mathcal{I}_{\mathbb{R}_+}}{r^2} \otimes \mathcal{L} = \frac{\partial^2}{\partial r^2} - \frac{n-1}{r} \frac{\partial}{\partial r} - \frac{(n-2)^2}{4r^2} + \frac{1}{r^2} \mathcal{L},$$

where  $\mathcal{I}_A$  is the identity operator on  $A$ . Consequently, it is natural to construct a null-sequence for  $P_\mu - W$  of the product form

$$\{\varphi_k(r, \omega)\}_{k=1}^\infty = \{u_k(r) \phi_k(\omega)\}_{k=1}^\infty$$

that converges locally uniformly to  $r^{(2-n)/2} \phi_\mu(\omega)$ , and by Theorem 2.4, this implies that the operator  $P_\mu - W$  is critical and  $r^{(2-n)/2} \phi_\mu(\omega)$  is its ground state.

Let  $\{u_k(r)\}_{k=1}^\infty$  be a null-sequence for the critical operator  $\mathcal{R}$  on  $\mathbb{R}_+$ , converging locally uniformly to  $r^{(2-n)/2}$ . So,

$$q_{\mathcal{R}}(u_k) \rightarrow 0, \quad \int_1^2 (u_k)^2 r^{n-1} dr = 1.$$

On the other hand, let  $\{\phi_k(\omega)\}_{k=1}^\infty$  be (up to the normalization constants) the sequence of ground states defined by (3.7) on  $\Sigma_k$ , so that

$$\int_{\Sigma} \phi_k^2 dS = 1, \quad \text{and } q_{\mathcal{L}}(\phi_k) = (\sigma_k(\mu) - \sigma(\mu)) \int_{\Sigma} \phi_k^2 dS \rightarrow 0.$$

Note that the normalization of  $\phi_k$  is different from the one used in the proof of Theorem 3.1. Recall that the operator  $\mathcal{L}_{\mu_0} = -\Delta_S - \mu_0 \delta_{\Omega}^{-2} - \sigma(\mu_0)$  is nonnegative on  $\Sigma$ . Therefore,

$$(5.4) \quad \frac{\mu\sigma(\mu_0)}{\mu_0} \int_{\Sigma} \phi_k^2 dS + \mu \int_{\Sigma} \frac{|\phi_k|^2}{\delta_{\Omega}^2} dS \leq \frac{\mu}{\mu_0} \int_{\Sigma} |\nabla_{\omega} \phi_k|^2 dS.$$

On the other hand,

$$(5.5) \quad \int_{\Sigma} |\nabla_{\omega} \phi_k|^2 dS = \sigma_k \int_{\Sigma} \phi_k^2 dS + \mu \int_{\Sigma} \frac{\phi_k^2}{\delta_{\Omega}^2} dS$$

By (5.4) and (5.5) we get

$$(5.6) \quad \left(1 - \frac{\mu}{\mu_0}\right) \int_{\Sigma} |\nabla_{\omega} \phi_k|^2 dS \leq \left(\sigma_k - \frac{\mu\sigma(\mu_0)}{\mu_0}\right) \int_{\Sigma} \phi_k^2 dS \leq \sigma_1 - \frac{\mu\sigma(\mu_0)}{\mu_0}$$

Since  $\mu < \mu_0$ , one gets that  $\{\phi_k\}$  is bounded in  $W_0^{1,2}(\Sigma)$ , and therefore (up to a subsequence),  $\{\phi_k\}$  converges, in  $L^2$  and locally uniformly to  $\phi$ , a positive solution of  $\mathcal{L}u = 0$  in  $\Sigma$  with  $\int_{\Sigma} \phi^2 dS = 1$ . Since  $\mathcal{L}$  is critical in  $\Sigma$ ,  $\phi = \phi_{\mu}$ . Hence, by the Harnack inequality,

$$\int_{\Sigma_1} \phi_k^2 dS \asymp 1,$$

and therefore  $\{\phi_k\}$  is a null-sequence.

We claim that there exists a subsequence  $\{k_l\} \subset \mathbb{N}$ , such that  $\{u_l(r)\phi_{k_l}(\omega)\}$  is a null-sequence for the operator  $P_{\mu} - W$  in  $\Omega$  that converges locally uniformly to  $r^{(2-n)/2}\phi_{\mu}(\omega)$ .

Indeed, fix the pre-compact open set  $B := \{(r, \omega) \mid r \in (1, 2), \omega \in \Sigma_1\}$ . Note that for the quadratic form  $Q$  of  $P_{\mu} - W$  in  $\Omega$ , if  $u = u(r)$  is compactly supported in  $\mathbb{R}_+$  and  $\psi = \psi(\omega)$  is compactly supported in  $\Sigma$ , we have

$$Q(u(r)\psi(\omega)) = q_{\mathcal{R}}(u) \|\psi\|_2^2 + \left(\int_0^\infty u^2(r)r^{n-3} dr\right) q_{\mathcal{L}}(\psi).$$

For each  $k$ , notice that by definition of a null-sequence,  $u_k$  is compactly supported in  $\mathbb{R}_+$ . So, for  $l \geq 1$ , let  $\{k_l\}_{l=1}^\infty$  be a subsequence such that

$$q_{\mathcal{R}}(u_l) \|\phi_{k_l}\|_2^2 = q_{\mathcal{R}}(u_l) < \frac{1}{l},$$

and

$$\left(\int_0^\infty u_l^2(r)r^{n-3} dr\right) q_{\mathcal{L}}(\phi_{k_l}) < \frac{1}{l}.$$

Thus,  $\lim_{l \rightarrow \infty} Q(u_l(r)\phi_{k_l}(\omega)) = 0$ .

On the other hand,  $\{u_l(r)\phi_{k_l}(\omega)\}$  converges uniformly in  $B$  to the function  $r^{(2-n)/2}\phi_{\mu}(\omega)$ , hence,  $\int_B (u_l(r)\phi_{k_l}(\omega))^2 dx \asymp 1$ .

Therefore,  $\{u_l(r)\phi_{k_l}(\omega)\}_{l=1}^\infty$  is indeed a null-sequence for  $P_\mu - W$ . It follows that  $P_\mu - W$  is critical in  $\Omega$  with the ground state  $r^{(2-n)/2}\phi_\mu(\omega)$ . Moreover, since  $\mathcal{R}$  is null critical around 0 and  $\infty$  it follows  $P_\mu - W$  is in fact null-critical around 0 and  $\infty$ .

Next we prove that the spectrum of  $W^{-1}P_\mu$  is  $[1, \infty)$ . Let us keep our assumption that  $\phi_\mu$  is normalized so that  $\|\phi_\mu\|_2 = 1$ . If  $\xi \in \mathbb{R}$ , then it easily checked (cf. [13]) that

$$\left(\mathcal{R} - \frac{(n-2)^2\xi^2}{4|x|^2}\right)(r^{n-2})^{i\xi-1/2} = 0,$$

therefore,

$$(5.7) \quad \left(P_\mu - \left(1 + \frac{(n-2)^2}{4\lambda(\mu)}\xi^2\right)W\right)\left((r^{n-2})^{i\xi-1/2}\phi_\mu(\omega)\right) = 0.$$

Define the subspace  $\mathcal{E}$  of  $L^2(\Omega, W \, dx)$  consisting of all functions of the form  $u(r)\phi_\mu(\omega)$ , where  $u \in L^2(\mathbb{R}_+, r^{n-1}\lambda(\mu)/r^2 \, dr)$ . We are going to define a spectral representation of  $W^{-1}P_\mu$  restricted to the subspace  $\mathcal{E}$ . Notice that the measure on  $\mathcal{E}$  is  $r^{n-1}\lambda(\mu)/(r^2) \, dr \otimes dS$ , so that

$$\mathcal{E} = L^2\left(\mathbb{R}_+, r^{n-1}\frac{\lambda(\mu)}{r^2} \, dr\right) \otimes \text{span}\{\phi_\mu\}.$$

Recall that the classical Mellin transform is the unitary operator  $\mathcal{M} : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R})$  defined by

$$\mathcal{M}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(r)r^{i\xi-1/2} \, dr.$$

Consider the composition  $\mathcal{C}$  of the unitary operator

$$\mathcal{U} : L^2\left(\mathbb{R}_+, r^{n-1}\frac{\lambda(\mu)}{r^2} \, dr\right) \rightarrow L^2(\mathbb{R}_+)$$

given by

$$f(r) \mapsto \sqrt{\frac{2\lambda(\mu)}{n-2}} f(r^{1/(n-2)}),$$

with the Mellin transform  $\mathcal{M}$ . Define

$$\mathcal{T} : \mathcal{E} \mapsto L^2(\mathbb{R}); \quad \mathcal{T}(u(r)\phi_\mu(\omega)) = (\mathcal{C}u)(\xi) = \left(\mathcal{M}(\mathcal{U}(u))\right)(\xi).$$

So,  $\mathcal{T}$  is a unitary operator. By (5.7), the operator  $\mathcal{T}(W^{-1}P_\mu)\mathcal{T}^{-1}$  is the multiplication by the real function  $(1 + (n-2)^2\xi^2/(4\lambda(\mu)))$  on  $L^2(\mathbb{R})$ , with values in  $[1, \infty)$ . Therefore, the spectrum of  $W^{-1}P_\mu$ , restricted to  $\mathcal{E}$  is  $[1, \infty)$ . So, the spectrum of  $W^{-1}P_\mu$  on  $L^2(\Omega, W \, dx)$  contains  $[1, \infty)$ . But the Hardy inequality (5.2) implies that the spectrum of  $W^{-1}P_\mu$  must be included in  $[1, \infty)$ . Hence, the spectrum of  $W^{-1}P_\mu$  on  $L^2(\Omega, W \, dx)$  is  $[1, \infty)$ .

For  $k \geq 2$ , define the subspace  $\mathcal{E}_k$  (resp.  $\mathcal{E}_{1/k}$ ) of  $L^2(\Omega, W \, dx)$  consisting of functions of the form  $u(r)\phi(\omega)$ , where  $u \in L^2((k, \infty), r^{n-1}\lambda(\mu)/r^2 \, dr)$  (resp.  $u \in L^2((0, 1/k), r^{n-1}\lambda(\mu)/r^2 \, dr)$ ). Denote by  $\mathcal{P}_k$  (resp.  $\mathcal{P}_{1/k}$ ) the restriction of  $P_\mu$  to  $\mathcal{E}_k$  (resp.  $\mathcal{E}_{1/k}$ ), with Dirichlet boundary conditions at  $\{k\} \times \Sigma$  (resp. at  $\{1/k\} \times \Sigma$ ). Notice that by symmetry considerations (under  $x \mapsto x^{-1}$ ), the spectrum of  $W^{-1}\mathcal{P}_k$  and the spectrum of  $W^{-1}\mathcal{P}_{1/k}$  are equal. Moreover, by the fact that the essential spectrum is stable under compactly supported perturbations, and since the discrete spectrum of  $W^{-1}P_\mu$  is empty, the spectrum of  $W^{-1}P_\mu$  is equal to the union of the spectrum of  $W^{-1}\mathcal{P}_k$ , and of the spectrum of  $W^{-1}\mathcal{P}_{1/k}$ . Thus, the spectra of  $W^{-1}\mathcal{P}_k$  and  $W^{-1}\mathcal{P}_{1/k}$  are both equal to  $[1, \infty)$ .

Also, the best constant  $C_0$  for the validity of the Hardy inequality

$$\int_{\mathcal{V}_0} \left( |\nabla \varphi|^2 - \frac{\mu}{\delta_\Omega^2} \varphi^2 \right) dx \geq C_0 \int_{\mathcal{V}_0} W \varphi^2 dx \quad \forall \varphi \in C_0^\infty(\mathcal{V}_0),$$

in  $\mathcal{V}_0$ , an arbitrarily small neighborhood of zero, is equal to the bottom of the essential spectrum of  $W^{-1}\mathcal{P}_{1/k}$  (for any  $k \geq 2$ ). Thus, it is equal to 1. Similarly, using  $W^{-1}\mathcal{P}_k$  instead, one concludes that the best constant  $C_\infty$  for the validity of the Hardy inequality

$$\int_{\mathcal{V}_\infty} \left( |\nabla \varphi|^2 - \frac{\mu}{\delta_\Omega^2} \varphi^2 \right) dx \geq C_\infty \int_{\mathcal{V}_\infty} W \varphi^2 dx \quad \forall \varphi \in C_0^\infty(\mathcal{V}_\infty),$$

in  $\mathcal{V}_\infty$ , an arbitrarily small neighborhood at infinity, is equal to 1. This finishes the proof of Theorem 5.4.  $\blacksquare$

We now turn to the case  $\mu = \mu_0$ , for which we need to assume more regularity on  $\Sigma$ .

**Theorem 5.6.** *Assume that  $\Sigma \in C^2$ .*

1. *If  $\mu_0 < 1/4$ , then  $\lambda(\mu_0) = 0$ , and the operator  $P_{\mu_0}$  is critical in  $\Omega$ , and null-critical around 0 and  $\infty$ . In particular, the Hardy inequality*

$$\int_{\Omega} |\nabla \varphi|^2 dx \geq \mu_0 \int_{\Omega} \frac{\varphi^2}{\delta_\Omega^2} dx \quad \forall \varphi \in C_0^\infty(\Omega),$$

*cannot be improved.*

2. *If  $\mu_0 = 1/4$  and  $\lambda(1/4) = 0$ , then the operator  $P_{1/4}$  is critical in  $\Omega$ , and null-critical around 0 and  $\infty$ . In particular, the Hardy inequality*

$$\int_{\Omega} |\nabla \varphi|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{\varphi^2}{\delta_\Omega^2} dx \quad \forall \varphi \in C_0^\infty(\Omega),$$

*cannot be improved.*

3. *If  $\mu_0 = 1/4$  and  $\lambda(1/4) > 0$ , then the weight  $W_{1/4} := \lambda(1/4)|x|^{-2}$  is optimal in the sense of Theorem 5.4. In particular, the Hardy inequality (5.2) cannot be improved. Moreover, The spectrum and the essential spectrum of the Friedrichs extension of the operator  $(W_{1/4})^{-1}P_{1/4}$  on  $L^2(\Omega, W_{1/4} dx)$  are both equal to  $[1, \infty)$ .*

*Proof.* Denote  $W(x) := \lambda(\mu_0)|x|^{-2}$ . Let us start by proving that in all cases,  $P_{\mu_0} - W$  is critical. Recall that in spherical coordinates  $P_{\mu_0} - W$  has the following skew-product form:

$$P_{\mu_0} - W = \mathcal{R} \otimes \mathcal{I}_\Sigma - \frac{\mathcal{I}_{\mathbb{R}_+}}{r^2} \otimes \mathcal{L} = \frac{\partial^2}{\partial r^2} - \frac{n-1}{r} \frac{\partial}{\partial r} - \frac{(n-2)^2}{4r^2} + \frac{1}{r^2} \mathcal{L}_{\mu_0}.$$

So, as in the first part of the proof of Theorem 5.4, it is natural to construct a null-sequence for  $P_{\mu_0} - W$  of the product form

$$\{\varphi_k(r, \omega)\}_{k=1}^\infty = \{u_k(r)\phi_k(\omega)\}_{k=1}^\infty$$

that converges locally uniformly to  $r^{(2-n)/2}\phi_{\mu_0}(\omega)$ .

As in the proof of Theorem 5.4, let  $\{u_k(r)\}_{k=1}^\infty$  be a null-sequence for the critical operator  $\mathcal{R}$  on  $\mathbb{R}_+$ , converging locally uniformly to  $r^{(2-n)/2}$ . So,

$$q_{\mathcal{R}}(u_k) \rightarrow 0, \quad \int_1^2 (u_k)^2 r^{n-1} dr = 1.$$

However, the definition of  $\{\phi_k\}$  differs from the one of Theorem 5.4. Let us normalize  $\phi_{\mu_0}$  so that  $\int_{\Sigma} \phi_{\mu_0}^2 dS = 1$  (by Lemma 3.4,  $\phi_{\mu} \in L^2(\Sigma)$ ). By lemmas 3.4 and 3.7, there exists a null-sequence  $\{\phi_k\}$  for  $\mathcal{L}_{\mu_0}$ , converging locally uniformly and in  $L^2(\Sigma)$  to  $\phi_{\mu_0}$ . Thus, normalizing  $\phi_k$  so that

$$\int_{\Sigma} \phi_k^2 dS = 1,$$

one has for  $k$  large enough, by the Harnack inequality,

$$\int_{\Sigma_1} \phi_k^2 dS \asymp 1.$$

Let  $B = \{(r, \omega) \mid r \in (1, 2), \omega \in \Sigma_1\}$ . We now choose the subsequence  $\{k_l\} \subset \mathbb{N}$  as in the proof of Theorem 5.4: let  $\{k_l\}_{l=1}^{\infty}$  be a subsequence such that

$$q_{\mathcal{R}}(u_l) \|\phi_{k_l}\|_2^2 = q_{\mathcal{R}}(u_l) < \frac{1}{l},$$

and

$$\left( \int_0^{\infty} u_l^2(r) r^{n-3} dr \right) q_{\mathcal{L}}(\phi_{k_l}) < \frac{1}{l}.$$

The same computation made in the proof of Theorem 5.4 shows that

$$\lim_{l \rightarrow \infty} Q(u_l(r) \phi_{k_l}(\omega)) = 0, \quad \text{and} \quad \int_B (u_l(r) \phi_{k_l}(\omega))^2 dx \asymp 1,$$

so that  $\{u_l(r) \phi_{k_l}(\omega)\}_{l=1}^{\infty}$  is indeed a null-sequence for  $P_{\mu} - W$ . It follows that  $P_{\mu} - W$  is critical in  $\Omega$  with a ground state  $r^{(2-n)/2} \phi_{\mu}(\omega)$ . Moreover, since  $\mathcal{R}$  is null critical around 0 and  $\infty$  it follows  $P_{\mu} - W$  is in fact null-critical around 0 and  $\infty$ .

1. Assume now that  $\mu_0 < 1/4$ . By the first part of the proof, the operator  $P_{\mu} - \lambda(\mu)|x|^{-2}$  is critical, and null-critical around 0 and  $\infty$ . By Lemma 3.4,  $\sigma(\mu_0) = -(n-2)^2/4$ , so  $\lambda(\mu_0) = 0$ . It follows that  $P_{\mu_0}$  is critical, and null-critical around 0 and  $\infty$ .

2. Suppose that  $\mu_0 = 1/4$ , and  $\lambda(1/4) = 0$ . Then by the first part of the proof, the operator  $P_{1/4} = P_{1/4} - \lambda(1/4)|x|^{-2}$  is critical, and null-critical around 0 and  $\infty$ .

3. Assume that  $\mu_0 = 1/4$ , and  $\lambda(1/4) > 0$ . Then following the proof of Theorem 5.4, one concludes that  $W$  is an optimal weight for  $P_{1/4}$ .  $\blacksquare$

In the particular case of the half-space we can compute the constants appearing in theorems 5.4 and 5.6.

**Example 5.7** (see [13, Example 11.9] and [18]). Let  $\Omega = \mathbb{R}_+^n$ ,  $\mu \leq \mu_0 = 1/4$  and consider the subcritical operator  $P_{\mu} := -\Delta - \mu x_1^{-2}$  in  $\Omega$ . Let  $\alpha_+$  be the largest root of the equation  $\alpha(1-\alpha) = \mu$ , and let

$$\eta(\mu) := n - 1 + \sqrt{1 - 4\mu} = n - 2 + 2\alpha_+.$$

Then

$$v_0(x) := x_1^{\alpha_+}, \quad v_1(x) := x_1^{\alpha_+} |x|^{-\eta(\mu)}$$

are two positive solutions of the equation  $P_{\mu} u = 0$  in  $\Omega$  that vanish on  $\partial\Omega \setminus \{0\}$ .

Therefore,  $\lambda(\mu) = \eta^2(\mu)/4$ , and for  $\mu \leq \mu_0 = 1/4$  we have the following optimal Hardy inequality

$$\int_{\mathbb{R}_+^n} |\nabla \varphi|^2 dx - \mu \int_{\mathbb{R}_+^n} \frac{\varphi^2}{x_1^2} dx \geq \frac{\eta^2(\mu)}{4} \int_{\mathbb{R}_+^n} \frac{\varphi^2}{|x|^2} dx \quad \forall \varphi \in C_0^{\infty}(\mathbb{R}_+^n).$$



In particular, the operator  $-\Delta - \mu x_1^{-2} - \lambda(\mu)|x|^{-2}$  is critical in  $\mathbb{R}_+^n$  with the ground state  $\psi(x) := x_1^{\alpha_+} |x|^{-\eta(\mu)/2}$ . Note that for  $\mu = 0$  we obtain the well known (optimal) Hardy inequality (see [27])

$$\int_{\mathbb{R}_+^n} |\nabla \varphi|^2 dx \geq \frac{n^2}{4} \int_{\mathbb{R}_+^n} \frac{\varphi^2}{|x|^2} dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}_+^n),$$

while for  $\mu = \mu_0 = 1/4$  we obtain the optimal double Hardy inequality (see [18])

$$(5.8) \quad \int_{\mathbb{R}_+^n} |\nabla \varphi|^2 dx - \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{1}{x_1^2} \varphi^2 dx \geq \frac{(n-1)^2}{4} \int_{\mathbb{R}_+^n} \frac{\varphi^2}{|x|^2} dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}_+^n).$$

It turns out that in the weakly mean convex case,  $\lambda(1/4)$  is always positive.

**Proposition 5.8.** *Assume that  $\Sigma \in C^2$  and  $\Omega$  is weakly mean convex. Then  $\lambda(1/4) > 0$ .*

*Proof.* Since  $\Omega$  is weakly mean convex (i.e.,  $-\Delta \delta_\Omega \geq 0$  in  $\Omega$ ), it follows that  $\delta_\Omega^{1/2}$  is a positive supersolution of  $P_{1/4} u = 0$  in  $\Omega$ . We proceed by contradiction: assume that  $\lambda(1/4) = 0$ . Then by Theorem 5.6 the operator  $P_{1/4}$  is critical and therefore  $\delta_\Omega^{1/2}$  is a positive solution of  $P_{1/4} u = 0$  in  $\Omega$ . Thus, necessarily  $-\Delta \delta_\Omega = 0$  in the sense of distributions. Since  $\delta_\Omega \in W_{\text{loc}}^{1,2}(\Omega)$  (or directly by Weyl's lemma) we have that  $\delta_\Omega$  is harmonic and in particular  $\delta_\Omega \in C^\infty(\Omega)$ . This means that the singular set of  $\delta_\Omega$ ,

$$\begin{aligned} \text{Sing}(\delta_\Omega) &:= \{x \in \Omega \mid \delta_\Omega(x) \text{ is achieved by more than one boundary points}\} \\ &= \{x \in \Omega \mid \delta_\Omega \text{ is not differentiable}\}, \end{aligned}$$

(see for example [15, Theorem 3.3]) is empty. In light of Motzkin theorem [35, Theorem 1.2.4],  $\mathbb{R}^n \setminus \Omega$  is convex. Since 0 is on the boundary of  $\mathbb{R}^n \setminus \Omega$ , by considering a supporting hyperplane of  $\mathbb{R}^n \setminus \Omega$  at 0, we find that necessarily  $\mathbb{R}^n \setminus \Omega$  is included in a half-space. This implies that  $\Sigma$  contains a half-sphere. If this half-sphere is strictly contained in  $\Sigma$ , then  $K := \mathbb{R}^n \setminus \Omega$  is a closed convex cone not containing a line (i.e.,  $K$  is *pointed*). Hence, its dual cone  $K^*$ , and thus its polar cone  $K^\circ = -K^* \subset \Omega$  has nonempty interior (see for instance [9, page 53]). Clearly,  $\delta_\Omega(x) = |x|$  whenever  $x \in K^\circ$ , but this contradicts the harmonicity of  $\delta_\Omega$  in  $\Omega$ .

Hence,  $\Sigma$  is precisely a half-sphere, and thus  $\Omega$  is a half-space. But by Example 5.7, in the half-space  $\{x_1 > 0\}$  we have  $\lambda(1/4) = (n-1)^2/4 > 0$ , and we arrived at a contradiction. ■

Assume that  $\Omega$  is a domain admitting a supporting hyperplane  $H$  at zero. Without loss of generality, we may assume that  $H = \partial \mathbb{R}_+^n$ . Recall that in this case  $\lambda_0(-\Delta, \delta_\Omega^{-2}, \Omega) \leq 1/4$  [24, Theorem 5]. Also,  $\delta_\Omega \leq \delta_H$  in  $\Omega$ . Consequently, for appropriate test functions  $\varphi_\varepsilon$  supported in a relative small neighborhood of the origin in  $\Omega$  we have that for  $0 \leq \mu \leq 1/4$  the corresponding Rayleigh-Ritz quotients satisfy the inequality

$$\frac{\int_\Omega (|\nabla \varphi_\varepsilon|^2 - \mu \frac{|\varphi_\varepsilon|^2}{\delta_\Omega^2}) dx}{\int_\Omega \frac{|\varphi_\varepsilon|^2}{|x|^2} dx} \leq \frac{\int_H (|\nabla \varphi_\varepsilon|^2 - \mu \frac{|\varphi_\varepsilon|^2}{\delta_H^2}) dx}{\int_H \frac{|\varphi_\varepsilon|^2}{|x|^2} dx} = \frac{(n-1 + \sqrt{1-4\mu})^2}{4} + o(1),$$

where  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Thus, Example 5.7 implies

**Corollary 5.9.** *Suppose that a domain  $\Omega$  admits a supporting hyperplane at zero, and let  $P_\mu = -\Delta - \mu \delta_\Omega^{-2}$ , where  $0 \leq \mu \leq 1/4$ . Then*

$$\lambda_0(P_\mu, |x|^{-2}, \Omega) \leq \frac{(n-1 + \sqrt{1-4\mu})^2}{4}.$$

## 6. ON THE OPTIMALITY OF AN INEQUALITY BY FILIPPAS, TERTIKAS AND TIDBLM

In the present section we generalize examples 1.4 and 5.7 concerning the half-space  $\mathbb{R}_+^n$ . We consider the following family of Hardy inequalities in  $\mathbb{R}_+^n$ , obtained by S. Filippas, A. Tertikas and J. Tidblom [18]:

$$(6.1) \quad \int_{\mathbb{R}_+^n} |\nabla \varphi|^2 dx \geq \int_{\mathbb{R}_+^n} \left( \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_1^2 + x_2^2} + \dots + \frac{\beta_n}{x_1^2 + \dots + x_n^2} \right) \varphi^2 dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}_+^n).$$

According to [18, Theorem A], the Hardy inequality (6.1) holds if and only if the  $\beta_i$ 's are of the following form:

$$(6.2) \quad \beta_1 = -\alpha_1^2 + \frac{1}{4}, \quad \beta_i = -\alpha_i^2 + \left( \alpha_{i-1} - \frac{1}{2} \right)^2 \quad i = 2, \dots, n,$$

where the  $\alpha_i$ 's are arbitrary real numbers. Without loss of generality, we can –and will– assume that all  $\alpha_i$ 's in (6.2) are nonpositive. Denote

$$V(\beta_1, \dots, \beta_j) = \left( \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_1^2 + x_2^2} + \dots + \frac{\beta_j}{x_1^2 + \dots + x_j^2} \right) \quad j = 1, \dots, n.$$

Let  $2^* = 2n/(n-2)$  be the Sobolev exponent. In [18, Theorem B], it is shown that (6.1) can be improved by adding to the right-hand side a Sobolev term of the form  $C(\int_{\mathbb{R}_+^n} |\varphi|^{2^*} dx)^{2/2^*}$  if and only if  $\alpha_n < 0$ . Notice that  $\beta_1, \dots, \beta_{n-1}$  being fixed, taking  $\alpha_n = 0$  corresponds to taking the greatest  $\beta_n$  possible in (6.2).

Our aim in this section is to show that when  $\alpha_n = 0$ , not only one cannot add a Sobolev term, but in fact one cannot even add any term of the form  $\int_{\mathbb{R}_+^n} W \varphi^2 dx$ ,  $W \geq 0$ , to the right hand side of (6.1). In other words, if  $\alpha_n = 0$ , the operator  $-\Delta - V(\beta_1, \dots, \beta_n)$  is *critical* in  $\mathbb{R}_+^n$ . This implies in particular (see [32]) that (6.1) cannot be improved by adding to the right-hand side any weighted Sobolev term of the form  $C(\int_{\mathbb{R}_+^n} \rho |\varphi|^{2^*} dx)^{2/2^*}$ , where  $\rho \geq 0$ ; an improvement of the result obtained in [18].

**Theorem 6.1.** *Consider the Hardy inequality (6.1), where the  $\beta_i$ 's are defined in term of non-positive  $\alpha_i$ 's by (6.2). Assume that  $\alpha_n = 0$ , and that  $\alpha_1, \dots, \alpha_{n-1}$  are either all distinct, or all negative. Then the operator  $P := -\Delta - V(\beta_1, \dots, \beta_n)$  is critical in  $\mathbb{R}_+^n$ , i.e., the Hardy inequality (6.1) cannot be improved. Furthermore, the weight  $\beta_n |x|^{-2}$  is an optimal weight for the subcritical operator  $-\Delta - V(\beta_1, \dots, \beta_{n-1})$  in  $\mathbb{R}_+^n$ .*

*Proof.* Denote  $X_k(x) := (x_1, \dots, x_k, 0, \dots, 0)$ . Let  $(\beta_i)_{i=1}^n$  satisfy (6.2), and define

$$\psi(x) := |X_1|^{-\gamma_1} |X_2|^{-\gamma_2} \dots |X_n|^{-\gamma_n},$$

where  $\gamma_i$  are defined by

$$\gamma_1 = \alpha_1 - \frac{1}{2}, \quad \gamma_i = \alpha_i - \alpha_{i-1} + \frac{1}{2} \quad i = 2, \dots, n.$$

Then,

$$\beta_1 = -\gamma_1(1 + \gamma_1), \quad \beta_i = -\gamma_i \left( 2 - i + \gamma_i + 2 \sum_{k=1}^{i-1} \gamma_k \right) \quad i = 2, \dots, n,$$

and according to equality (2.3) in [18],

$$-\frac{\Delta \psi}{\psi} = V(\beta_1, \dots, \beta_n).$$

Hence,  $\psi$  is a positive solution of the equation  $Pu = 0$  in  $\mathbb{R}_+^n$ . By the AAP Theorem, this implies the validity of (6.1).

For  $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n \setminus \{0\}$ , denote

$$r = |x|, \quad \omega = \frac{x}{|x|}, \quad \omega_i = \frac{x_i}{r} \quad 1 \leq i \leq n.$$

Notice that  $\omega \in \mathbb{S}_+ := \mathbb{S}^{n-1} \cap \{x_1 > 0\}$ . Since  $\alpha_n = 0$  we have

$$\psi(x) = \phi(\omega)r^{-\sum_{i=1}^n \gamma_i} = \phi(\omega)r^{(2-n)/2},$$

where

$$\phi(\omega) := \psi|_{\mathbb{S}_+} = \omega_1^{-\gamma_1} (\omega_1^2 + \omega_2^2)^{-\gamma_2/2} \dots (\omega_1^2 + \dots + \omega_n^2)^{-\gamma_n/2}.$$

Define

$$W(\omega) := \frac{\beta_1}{\omega_1^2} + \dots + \frac{\beta_{n-1}}{\omega_1^2 + \dots + \omega_{n-1}^2},$$

and let

$$\mathcal{L} := -\Delta_{\mathbb{S}^{n-1}} - W(\omega) - \beta_n + \frac{(n-2)^2}{4}, \quad \text{and} \quad \mathcal{R} := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r} \frac{\partial}{\partial r} - \frac{(n-2)^2}{4r^2}.$$

Then, in spherical coordinates,  $P$  has the skew-product form

$$P = \mathcal{R} + \frac{1}{r^2} \mathcal{L}.$$

Recall that  $\mathcal{R}$  is critical on  $(0, \infty)$ , and its ground state is  $r^{(2-n)/2}$ .

**Lemma 6.2.** *The operator  $\mathcal{L}$  is critical on  $\mathbb{S}_+$ , with ground state  $\phi \in L^2(\mathbb{S}_+)$ .*

Once Lemma 6.2 is proved, the rest of the proof of Theorem 6.1 follows along the lines of the proof of Theorem 5.4. ■

*Proof of Lemma 6.2.* We have

$$P\psi = 0 = \phi \mathcal{R} r^{(2-n)/2} + r^{-(n+2)/2} \mathcal{L}\phi.$$

Since

$$\mathcal{R} r^{-(n-2)/2} = 0 \quad \text{in } \mathbb{R}_+,$$

one concludes that

$$\mathcal{L}\phi = 0 \quad \text{in } \mathbb{S}_+.$$

For  $x \in \mathbb{S}_+$ , let  $\rho$  be the spherical distance function to  $\partial\mathbb{S}_+ = \{\omega \in \mathbb{S}_+ \mid \omega_1 = 0\}$ , the boundary of  $\mathbb{S}_+$ . Let  $dS$  be the Riemannian measure on  $\mathbb{S}_+$ . We claim that

$$(6.3) \quad \int_{\mathbb{S}_+ \cap \{\rho \leq \frac{1}{2}\}} \left( \frac{\phi(\omega)}{\rho \log(\rho)} \right)^2 dS < \infty.$$

Clearly, (6.3) implies that  $\phi \in L^2(\mathbb{S}_+)$ , and moreover, by Lemma 3.7, (6.3) implies that  $\mathcal{L}$  is critical with the ground state  $\phi$ . In fact, since  $\phi$  is smooth in the interior of  $\mathbb{S}_+$ , and

$$\rho(\omega) \sim \omega_1(\omega) \quad \text{as } \omega \in \mathbb{S}_+, \quad \text{and } \rho(\omega) \rightarrow 0,$$

(6.3) is equivalent to

$$(6.4) \quad \int_{\mathbb{S}_+ \cap \{\omega_1 \leq \frac{1}{2}\}} \left( \frac{\phi(\omega)}{\omega_1 \log(\omega_1)} \right)^2 dS < \infty.$$

For  $i = 1, \dots, n-1$ , define

$$\mathcal{E}_i = \{\omega \in \mathbb{S}_+ \mid \omega_1 \leq \varepsilon, \dots, \omega_i^2 \leq \varepsilon, \omega_{i+1}^2 > \varepsilon\}.$$

Then, all the  $\mathcal{E}_i$  are disjoint, and if  $\varepsilon < 1/n$ , one can write the  $\varepsilon$ -neighborhood  $\mathbb{S}_+ \cap \{\omega_1 \leq \varepsilon\}$  of  $\partial\mathbb{S}_+$  as the disjoint union:

$$\mathbb{S}_+ \cap \{\omega_1 \leq \varepsilon\} = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_{n-1}.$$

Notice that on  $\mathcal{E}_i$ ,

$$\phi(\omega) \leq C_\varepsilon \omega_1^{-\gamma_1} (\omega_1^2 + \omega_2^2)^{-\gamma_2/2} \dots (\omega_1^2 + \dots + \omega_i^2)^{-\gamma_i/2}.$$

Hence,

$$\int_{\mathcal{E}_i} \left( \frac{\phi(\omega)}{\omega_1 \log(\omega_1)} \right)^2 dS \leq C_\varepsilon \int_{\mathcal{E}_i} \log^{-2}(\omega_1) \omega_1^{-2} \omega_1^{-2\gamma_1} \dots (\omega_1^2 + \dots + \omega_i^2)^{-\gamma_i} dS.$$

If  $\varepsilon$  is small enough, then on  $\mathcal{E}_i$ ,

$$dS \simeq d\omega_1 \otimes \dots \otimes d\omega_i \otimes d\nu(\omega_1, \dots, \omega_i),$$

where  $d\nu(\omega_1, \dots, \omega_i)$  is the standard Hausdorff measure on the  $n-i-1$ -sphere  $\omega_{i+1}^2 + \dots + \omega_n^2 = \sigma^2$ , with  $\sigma^2 = 1 - (\omega_1^2 + \dots + \omega_i^2)$ . Thus,

$$(6.5) \quad \int_{\mathcal{E}_i} \left( \frac{\phi(\omega)}{\omega_1 \log(\omega_1)} \right)^2 dS \leq \tilde{C}_\varepsilon \int_{[0, \varepsilon]^i} \log^{-2}(\omega_1) \omega_1^{-2} \omega_1^{-2\gamma_1} \dots (\omega_1^2 + \dots + \omega_i^2)^{-\gamma_i} d\omega_1 \dots d\omega_i.$$

For  $\lambda_1, \dots, \lambda_i$  real numbers and  $k$  integer, define

$$I_i(\lambda_1, \dots, \lambda_i, k) := \int_{[0, \varepsilon]^i} \log^{-2}(\omega_1) \omega_1^{-2} \omega_1^{-2\lambda_1} \dots (\omega_1^2 + \dots + \omega_i^2)^{-\lambda_i} |\log^k(\omega_1^2 + \dots + \omega_i^2)| d\omega_1 \dots d\omega_i.$$

One has the elementary fact:

$$(6.6) \quad I_i(\lambda_1, \dots, \lambda_i, k) \leq C_\varepsilon \begin{cases} I_{i-1}(\lambda_1, \dots, \lambda_{i-2}, \lambda_{i-1} + \lambda_i - 1/2, k), & \lambda_i > 1/2, \\ I_{i-1}(\lambda_1, \dots, \lambda_{i-1}, k), & \lambda_i < 1/2, \\ I_{i-1}(\lambda_1, \dots, \lambda_{i-1}, k+1), & \lambda_i = 1/2. \end{cases}$$

*Case 1: assume that the  $\alpha_k$ 's,  $k = 1, \dots, n-1$ , are all distinct.* Then, for every  $2 \leq j \leq k \leq i$ ,

$$\gamma_j + \sum_{l=j+1}^k \left( \gamma_l - \frac{1}{2} \right) = \alpha_k - \alpha_{j-1} + \frac{1}{2} \neq \frac{1}{2}.$$

Moreover,

$$(6.7) \quad -2 - 2\gamma_1 - 2 \sum_{j=2}^k \left( \gamma_j - \frac{1}{2} \right) = -2 - 2\alpha_k - (k-2) + (k-1) = -2\alpha_k - 1.$$

Thus, by using (6.6)  $i$ -times in (6.5), and (6.7), one gets

$$\begin{aligned} \int_{\mathcal{E}_i} \left( \frac{\phi(\omega)}{\omega_1 \log(\omega_1)} \right)^2 dS &\leq C \sum_{k=1}^i \int_0^\varepsilon \log(\omega_1)^{-2} \omega_1^{-2-2\gamma_1-2\sum_{j=2}^k(\gamma_j-1/2)} d\omega_1 \\ &\leq C \sum_{k=1}^i \int_0^\varepsilon \log(\omega_1)^{-2} \omega_1^{-2\alpha_k-1} d\omega_1, \end{aligned}$$

where by convention the sum  $\sum_{j=2}^k$  is zero when  $k = 1$ . By hypothesis,  $\alpha_k \leq 0$ , therefore  $\log(\omega_1)^{-2} \omega_1^{-2\alpha_k-1}$  is integrable at zero, and thus one concludes the validity of (6.3).

Case 2: assume that  $\alpha_k < 0$ , for all  $k = 1, \dots, n-1$ . Then, by using (6.6)  $i$ -times in (6.5), and (6.7), one gets

$$\begin{aligned} \int_{\mathcal{E}_i} \left( \frac{\phi(\omega)}{\omega_1 \log(\omega_1)} \right)^2 dS &\leq C \sum_{k=1}^i \int_0^\varepsilon |\log^{n(k)}(\omega_1)| \omega_1^{-2-2\gamma_1-2\sum_{j=2}^k(\gamma_j-1/2)} d\omega_1 \\ &\leq C \sum_{k=1}^i \int_0^\varepsilon |\log^{n(k)}(\omega_1)| \omega_1^{-2\alpha_k-1} d\omega_1, \end{aligned}$$

where  $n(k)$  is an integer. Since  $\alpha_k < 0$ , the function  $|\log^{n(k)}(\omega_1)| \omega_1^{-2\alpha_k-1}$  is integrable at zero, and therefore (6.3) holds.  $\blacksquare$

**Remark 6.3.** We believe that Theorem 5.6 should hold in the general case, without any extra assumption on  $\alpha_1, \dots, \alpha_{n-1}$ . We leave this question for a future investigation.

## 7. A DIFFERENTIAL INEQUALITY

Throughout the present section,  $\Omega$  denotes a domain in  $\mathbb{R}^n$  such that  $0 \in \partial\Omega$ , and  $P_\mu = -\Delta - \mu\delta_\Omega^{-2}$ . Our aim is to obtain a Hardy-type inequality with the best constant for the (nonnegative) operator  $P_\mu$  in  $\Omega$ , assuming that  $\delta_\Omega$  satisfies the linear differential inequality

$$(7.1) \quad -\Delta\delta_\Omega + \frac{n-1+\sqrt{1-4\mu}}{|x|^2} (x \cdot \nabla\delta_\Omega - \delta_\Omega) \geq 0 \quad \text{in } \Omega.$$

The above differential inequality certainly holds true for any  $\mu \leq 1/4$  if  $\Omega$  is a weakly mean convex cone (see Definition 2.5); it also holds for  $\mu = 1/4$  if  $\Omega$  is a ball touching the origin (see Remark 7.2).

For  $\mu = 1/4$ , (7.1) is equivalent to the differential inequality

$$-|x|^{n-1} \operatorname{div}(|x|^{1-n} \nabla\delta_\Omega) - \frac{n-1}{|x|^2} \delta_\Omega \geq 0 \quad \text{in } \Omega.$$

It is worth mentioning here that in [17, Theorem 3.2] S. Filippas, L. Moschini, and A. Tertikas obtain improved Hardy inequality under the assumption that  $\Omega$  is a *bounded* domain such that  $0 \in \Omega$ , and  $\delta_\Omega$  satisfies the differential inequality

$$-\operatorname{div}(|x|^{2-n} \nabla\delta_\Omega) \geq 0 \quad \text{in } \Omega,$$

while K.T. Gkikas in [19] proves the Hardy inequality in an *exterior* domain  $\Omega$  such that  $0 \in \mathbb{R}^n \setminus \bar{\Omega}$ , and  $\delta_\Omega$  satisfies the differential inequality

$$-\operatorname{div}(|x|^{1-n} \nabla\delta_\Omega) \geq 0 \quad \text{in } \Omega.$$

Let

$$(7.2) \quad \eta(\mu) := n-1 + \sqrt{1-4\mu}.$$

Recall that for  $\Omega = \mathbb{R}_+^n$ , we obtained in Example 5.7 that  $\lambda_0(P_\mu, |x|^{-2}, \Omega) = \eta^2(\mu)/4$ . The following theorem shows that if  $\Omega$  is a domain such that  $\delta_\Omega$  is a positive supersolution of a certain second-order linear elliptic equation, then  $\lambda_0(P_\mu, |x|^{-2}, \Omega) \geq \eta^2(\mu)/4$ .

**Theorem 7.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$  such that  $0 \in \partial\Omega$ . Fix  $\mu \leq 1/4$ , and let  $\eta(\mu)$  be as in (7.2). Suppose that  $\delta_\Omega$  satisfies the following differential inequality*

$$(7.3) \quad -\Delta\delta_\Omega + \frac{\eta(\mu)}{|x|^2} (x \cdot \nabla\delta_\Omega - \delta_\Omega) \geq 0 \quad \text{in } \Omega$$

in the sense of distributions. Then the following improved Hardy inequality holds

$$(7.4) \quad \int_{\Omega} |\nabla \varphi|^2 dx - \mu \int_{\Omega} \frac{|\varphi|^2}{\delta_{\Omega}^2} dx \geq \frac{\eta^2(\mu)}{4} \int_{\Omega} \frac{|\varphi|^2}{|x|^2} dx \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

Assume further that  $\Omega$  admits a supporting hyperplane at zero and  $\mu \geq 0$ , then

$$\lambda_0(P_{\mu}, |x|^{-2}, \Omega) = \frac{\eta^2(\mu)}{4}.$$

*Proof.* As in Example 5.7, we write  $\alpha_+$  for the largest root of the equation  $\alpha(1 - \alpha) = \mu$ , and  $\psi := \delta_{\Omega}^{\alpha_+} |x|^{-\eta(\mu)/2}$ . We will show that  $\psi$  is a supersolution of the equation

$$(P_{\mu} - (\eta(\mu)/2)^2 |x|^{-2}) u = 0 \quad \text{in } \Omega,$$

and then (7.4) follows from the AAP theorem (Theorem 2.1). By direct computations we get

$$\begin{aligned} & \left( P_{\mu} - \frac{\eta^2(\mu)}{4|x|^2} \right) \psi \\ &= \alpha_+ \left( -\Delta \delta_{\Omega} + \frac{\eta(\mu)}{|x|^2} x \cdot \nabla \delta_{\Omega} \right) \delta_{\Omega}^{\alpha_+ - 1} |x|^{-\eta(\mu)/2} + \frac{\eta(\mu)}{2} (n - 2 - \eta(\mu)) \delta_{\Omega}^{\alpha_+} |x|^{-\eta(\mu)/2 - 2} \\ &= \alpha_+ \left( -\Delta \delta_{\Omega} + \frac{\eta(\mu)}{|x|^2} (x \cdot \nabla \delta_{\Omega} - \delta_{\Omega}) \right) \geq 0, \end{aligned}$$

where for the second equality we have used the fact that  $n - 2 - \eta(\mu) = -2\alpha_+$ , which follows from our choice of  $\alpha_+$ .

Assume that  $\Omega$  is a domain admitting a supporting hyperplane  $H$  at zero. Without loss of generality, we may assume that  $H = \partial \mathbb{R}_+^n$ . Then by Corollary 5.9 we have that  $\lambda_0(P_{\mu}, |x|^{-2}, \Omega) \leq \eta^2(\mu)/4$ . Thus,  $\lambda_0(P_{\mu}, |x|^{-2}, \Omega) = \eta^2(\mu)/4$ .  $\blacksquare$

**Remark 7.2.** 1. By (2.4), inequality (7.3) holds true for any  $\mu \leq 1/4$  if  $\Omega$  is a weakly mean convex cone.

We claim that (7.3) holds true also for  $\mu = 1/4$  in any ball  $B$  with  $0 \in \partial B$ , and consequently, the Hardy inequality (7.4) is valid in this case.

Indeed, let  $B = B_R(x_0)$  be an open ball in  $\mathbb{R}^n$  centered at  $x_0$ , such that  $|x_0| = R$ . Then for  $x \in B$  we have  $\delta_B(x) = R - |x_0 - x|$ , and simple computations show that for any  $x \in B \setminus \{x_0\}$

$$\nabla \delta_B(x) = \frac{x_0 - x}{|x_0 - x|} \quad \text{and} \quad -\Delta \delta_B(x) = \frac{n-1}{|x_0 - x|}.$$

Thus, for (7.3) to be true it is enough that for any  $x \in B \setminus \{x_0\}$  we have

$$-\Delta \delta_B + \frac{\eta(\mu)}{|x|^2} (x \cdot \nabla \delta_B - \delta_B) = \frac{n-1}{|x_0 - x|} + \frac{n-1}{|x|^2} \left( x \cdot \frac{(x_0 - x)}{|x_0 - x|} - R + |x_0 - x| \right) \geq 0.$$

After some cancelations this is equivalent to

$$(7.5) \quad |x|^2 \geq (R|x_0 - x| - x_0 \cdot (x_0 - x)) \quad \forall x \in B.$$

Some further simple computations implies that (7.5) is equivalent to

$$(x_0 - x) \cdot x \leq R^2 - R|x_0 - x| \quad \forall x \in B.$$

This is true since

$$2(x_0 - x) \cdot x = R^2 - |x|^2 - |x_0 - x|^2 \leq R^2 - |x_0 - x|^2 \leq 2(R^2 - R|x_0 - x|),$$

where in the last inequality we have used  $\alpha^2 - \beta^2 \leq 2(\alpha^2 - \alpha\beta)$  for all  $\alpha, \beta \in \mathbb{R}$ .

2. If the origin is an isolated point of  $\partial\Omega$ . Then the classical Hardy inequality near 0 and Theorem 2.1 imply that inequality (7.3) cannot hold.
3. It would be interesting to characterize the domains for which (7.3) hold true.

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